Quantum correlations
from entanglement to discord

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Overview

- Quantum correlations: a brief history
- Quantum entanglement & quantum discord
- Quantum discord & condensed matter
- Quantum discord & quantum optics
- Next
The relevance of quantum correlations

**Quantum correlations**: correlations typical of quantum systems, that cannot be explained by classical physics

Crucial for **theory**…

- understanding quantum mechanics from **fundamental perspective**: quantum nonlocality, measurement theory, decoherence
- understanding quantum behavior reflected in the **properties of matter**

…and **applications**!

- they are the **essential resource** for quantum technology: quantum communication, quantum computation, quantum metrology
Quantum correlations: a very brief history

- **1920es**: birth of quantum mechanics

- **1950es**: can we explain quantum mechanics in classical terms? Quantum correlations are recognized as a key concept to address this question

- **1960es**: research on entangled particles reveals quantum nonlocality: quantum physics is fundamentally different from classical physics

- **1980es**: correlations between quantum objects and their environment can explain the emergence of "classical reality" from the underlying quantum world
Quantum correlations: a very brief history

- **1990es**: rise of quantum information theory: realization that quantum correlations open the door to novel communicational and computational tasks (quantum computation, quantum teleportation, quantum dense coding...)

- **2000es**: entanglement and condensed matter theory: analysis of quantum correlations essential in characterizing and simulating quantum matter

- **2010es**: entanglement does not account for all quantum correlations. New concept of quantum discord. Research on role of discord in quantum science and technology
Quantum mechanics: recap

- Quantum mechanics is about the probabilities of obtaining measurement results, given that the system has been initialized ("prepared") in some way.

- The possible pure states of a quantum system form a vector space, the Hilbert space $\mathcal{H}$.

- A pure state $|\Psi\rangle$ represents the finest possible preparation of a quantum system (maximal information about the system).

- In general, the preparation is imperfect: the "mixed" state of the system is described by a density matrix $\varrho \in E(\mathcal{H})$.

- $\varrho$ represents a statistical mixture of states: $\varrho = \sum_j p_j |\Psi_j\rangle \langle \Psi_j|$, i.e., a sort of probability density over the Hilbert space.
Quantum mechanics: recap

- A property of a system is represented a subspace of the Hilbert space $S \subset \mathcal{H}$ (set of states satisfying the property).
- A subspace $S$ is specified by a projection operator $\Pi^S$ on that subspace.
- An "observable" is represented by an operator $\mathcal{A} = \sum_i a_i \Pi^i$ such that $\Pi^i = |i\rangle\langle i|$ are projectors onto the subset corresponding to $a_i$.
- Given a state $\rho$ and an observable $\mathcal{A}$:
  - the probability of measuring $a_i$ is given by $p_i = \text{Tr}[\rho \Pi^i]$ that represents the overlap between the state $\rho$ and the subset $S_i$ corresponding to $a_i$.
  - the post-measurement state corresponding to measurement result $a_i$ is $\tilde{\rho}^i = \frac{1}{p_i} [\Pi^i \rho \Pi^i]$ that represents the projection of $\rho$ onto $S_i$.
  - The average post-measurement state is $\tilde{\rho} = \sum_i [\Pi^i \rho \Pi^i]$. 
to define correlations, we need a definition of *subsystems*

given two systems with Hilbert space \( \mathcal{H}_A \) and \( \mathcal{H}_B \), the joint system has Hilbert space \( \mathcal{H}_A \otimes \mathcal{H}_B \)

Conversely, given a general Hilbert space \( \mathcal{H} \cong \mathbb{C}^n \), it can be ”divided” in several ways: for instance
\[
\mathbb{C}^8 \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \text{ and } \mathbb{C}^8 \cong \mathbb{C}^2 \otimes \mathbb{C}^4.
\]

The jargon is that there are several possible tensor product structures (TPS) over \( \mathcal{H} \): TPS define quantum subsystems
Correlations

- Given a state $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$, the ("reduced") states of the subsystems are: $\rho_A = \text{Tr}_B \rho$, $\rho_B = \text{Tr}_A \rho$
- The system is uncorrelated when $\rho = \rho_A \otimes \rho_B$
- The system is correlated when $\rho \neq \rho_A \otimes \rho_B$
- This means that each subsystem has information about the other
- General measure of correlations: quantum mutual information $I = S(\rho_A) + S(\rho_B) - S(\rho)$
- von Neumann entropy $S(\rho) = \text{Tr} \rho \log \rho$: measures how broadly $\rho$ is spread over the Hilbert space
- For correlated states $I > 0$: the global state is known with more accuracy than one of the single parts
Quantum correlations: quantum entanglement

- **Quantum systems can contain ”nonlocal” correlations**
- joint system composed of subsystems $A$ and $B$
- suppose we prepare states starting from an uncorrelated state through a sequence of local operations $\mathcal{O}_A \otimes \mathcal{O}_B$, operating separately on either subsystem
- then the states must contain no ”nonlocal” correlations
- such states are called *separable*
- we can prove that a state $\rho$ is separable if and only if it can be written as a mixture of product states:

$$\rho = \sum_j p_j \rho_A^j \otimes \rho_B^j$$

- (in the special case of pure state, a state is separable if and only if it can be written as a product: $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$)
Quantum correlations: quantum entanglement

- in the general case,

$$\rho \neq \sum_j p_j \rho_A^j \otimes \rho_B^j$$

- These states must be prepared with global (nonlocal) operations. They are called *entangled* states.

- Entangled states display strong correlations that are impossible in classical mechanics (and indeed can violate, e.g., Bell’s inequality).
The relevance of entanglement

- foundations of quantum mechanics: entanglement necessary condition for nonlocality (violation of Bell’s inequalities)
- quantum computation: entanglement necessary condition for speedup in pure state quantum computation.
- quantum communication: entanglement relevant for protocols like quantum teleportation, quantum dense coding, quantum key distribution
- condensed matter: study of entanglement essential in understanding and designing new efficient simulation strategies for many-body systems at low T
Quantum correlations: quantum discord

- Entanglement is not the only kind of genuinely "quantum" correlation
- *Quantum systems can be (globally) disturbed by (local) measurements*
- If a measurement on a single subsystem alters the global correlations, then the subsystems are quantumly correlated
- this is a quantum feature that has no classical counterpart (classically, correlations remain unaltered by measurements on a subsystem)
Quantum correlations: quantum discord

- joint system composed of subsystems $A$ and $B$
- perform local measurement $\Pi_B$ on subsystem $B$
- The average post-measurement state is
  \[
  \tilde{\rho} = \sum_i \Pi_B^i \rho \Pi_B^i
  \]
  with $p_i = \text{Tr}[\Pi_B^i \rho \Pi_B^i]$ being the probability of outcome $i$.
- In general, the measurement of the $B$ disrupts correlations between $A$ and $B$: $I(\tilde{\rho}) \leq I(\rho)$
Quantum discord measures the *minimal* amount of correlations that have been lost: \( Q = \min_{\Pi} (I(\rho) - I(\tilde{\rho})) \)

The optimization is over all possible measurements \( \Pi \)

In general, it is very difficult to identify the optimal measurement disturbs the systems - and correlations - the least; this makes a nontrivial technical issue in computing discord

For pure state, entanglement and discord coincide

For mixed states, entanglement and discord are different: in particular, separable states can have nonzero discord and, hence, contain quantum correlations that are not in the form of entanglement.
The relevance of quantum discord

- foundations of quantum mechanics: better understanding the quantum-classical difference

- quantum discord may allow for quantum computation, quantum communication and quantum metrology with mixed states that are not entangled

- entanglement is rapidly destroyed by noise, discord is often more robust
Quantum discord in condensed matter

Why to investigate quantum discord in condensed matter systems:

- Do condensed matter systems contain a significant amount of quantum correlations in the form of quantum discord?
- Can quantum discord be related/explain some physical properties of these systems?
- Conversely, can these systems illuminate general properties of discord?
Quantum discord in the extended Hubbard model


- Quantum discord in the 1-D extended Hubbard model

- Simplest model of strongly correlated electrons:
  \[ H = H_{\text{hopping}} + H_{\text{filling}} + H_{\text{on-site repulsion}} \]

  \[ H_{\text{hopping}} = \sum_{\langle i,j \rangle, \sigma} [1 - x(n_{i\sigma} + n_{i\bar{\sigma}})] c_{i\sigma}^{\dagger} c_{i\sigma} \]
  \[ H_{\text{filling}} = -\mu \sum_{i\sigma} n_{i\sigma} \]
  \[ H_{\text{on-site repulsion}} = u/2 \sum_{i\sigma} (n_{i\sigma} - 1/2)(n_{i\bar{\sigma}} - 1/2) \]
Quantum discord in the extended Hubbard model

- Exactly solvable model (simpler version of Hubbard model)

- The ground state appears as follows:

\[ |GS\rangle = \left( \sum_{k>K_s} c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \right)^{N_d} \prod_{|k|<K_s} c_k^\dagger |0\rangle \]

- \( \eta \)-pairs are spinless fermions

- \( \eta \)-pairs are responsible for the appearance of off-diagonal long-range order (ODLRO), a kind of long-range correlation that is at the basis of superconductivity
Quantum discord in the extended Hubbard model

The model has a nontrivial phase diagram when the chemical potential $\mu$ and the repulsion strength $u$ vary

- Some phases (II and III) are characterized by the presence of $\eta$-pairs, hence of ODLRO
Quantum discord in the extended Hubbard model

- We evaluate quantum discord in all phases of the model (we use novel technique to evaluate discord for qutrits).
- We evaluate the scaling of discord in the vicinity of the quantum phase transitions.
- We compare the behavior of entanglement and discord in the system.
Quantum discord in the extended Hubbard model

- Some phases have discord in absence of entanglement
- We are able to relate ODLRO, which is at the basis of superconductivity, to a measure of quantum correlations, quantum discord
- We highlight a general property of discord: it violates the monogamy property (i.e., a subsystem can be correlated with arbitrarily many other subsystems), the maximal violation being in the presence of ODLRO
- We better characterize the physics of the system close to the critical lines, in particular the appearance/disappearance of ODLRO
Why to investigate quantum discord in quantum optical systems:

- Do photonic states contain a significant amount of quantum correlations in the form of quantum discord? Which states have maximal correlations?

- What are the measurements that disturb the system minimally? Can they be carried over with realistic setups?

- Can we create quantum discord with experimentally realistic resources?
A paradigm in quantum optics is to consider Gaussian states.

Gaussian states can be prepared using only linear optical devices (beam splitters, parametric amplifiers, etc.)

They have a Gaussian statistics.

All information about the state is in correlation matrix $\sigma$, whose entries are quadratic correlation functions $\langle a_ia_j \rangle$, $\langle a_i^\dagger a_j \rangle$, etc.

Goal: evaluate quantum discord in Gaussian states.

Problem: which are the measurements that disturb the system the least?
it is easy to compute the effect of Gaussian measurements, i.e. measurements that preserve Gaussianity. The typical Gaussian measurement is homodyne detection.

optimizing only over Gaussian measurements, analytical formula for discord was obtained

\[
D^G(A : B) = h(\sqrt{\det \sigma_B}) - h(d_-) - h(d_+) + \min_{\sigma_P} h(\sqrt{\det \sigma_P})
\]

Question: are there non-Gaussian measurements that are better (they are less disturbing) than Gaussian ones? (The typical non-Gaussian measurement is photon counting)
We consider a large class of Gaussian states, mixed thermal states (MTS) and squeezed thermal states (STS) and compare the effect of Gaussian and non-Gaussian measurements.

For the large class of states analyzed, in the range of parameters considered, the Gaussian measurements are optimal.

We have thus strong evidence that the Gaussian measurements are always optimal for Gaussian states.

This might be verified experimentally using photon counters and homodyne detection.
Next: quantum correlations in complex quantum fermionic networks

- Networks of quasi-free fermions: \( H = \sum_{ij=1}^{N} A_{ij} c_i^\dagger c_j \)
- \( A \) is the adjacency matrix of a general graph (not a regular lattice)

Goal: a general study of quantum correlations (entanglement and discord) in these models
- How do the topological properties of the network affect correlations?
- Which nodes are the most correlated?
- How does the nature of multipartite/bipartite correlations depend on the density of links in the network?
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Publications 2011-2012: