Infrared divergences of gauge theory scattering amplitudes

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Seminario Scuola di dottorato in scienze della natura e tecnologie innovative
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Outline:

- Introduction to the infrared divergences

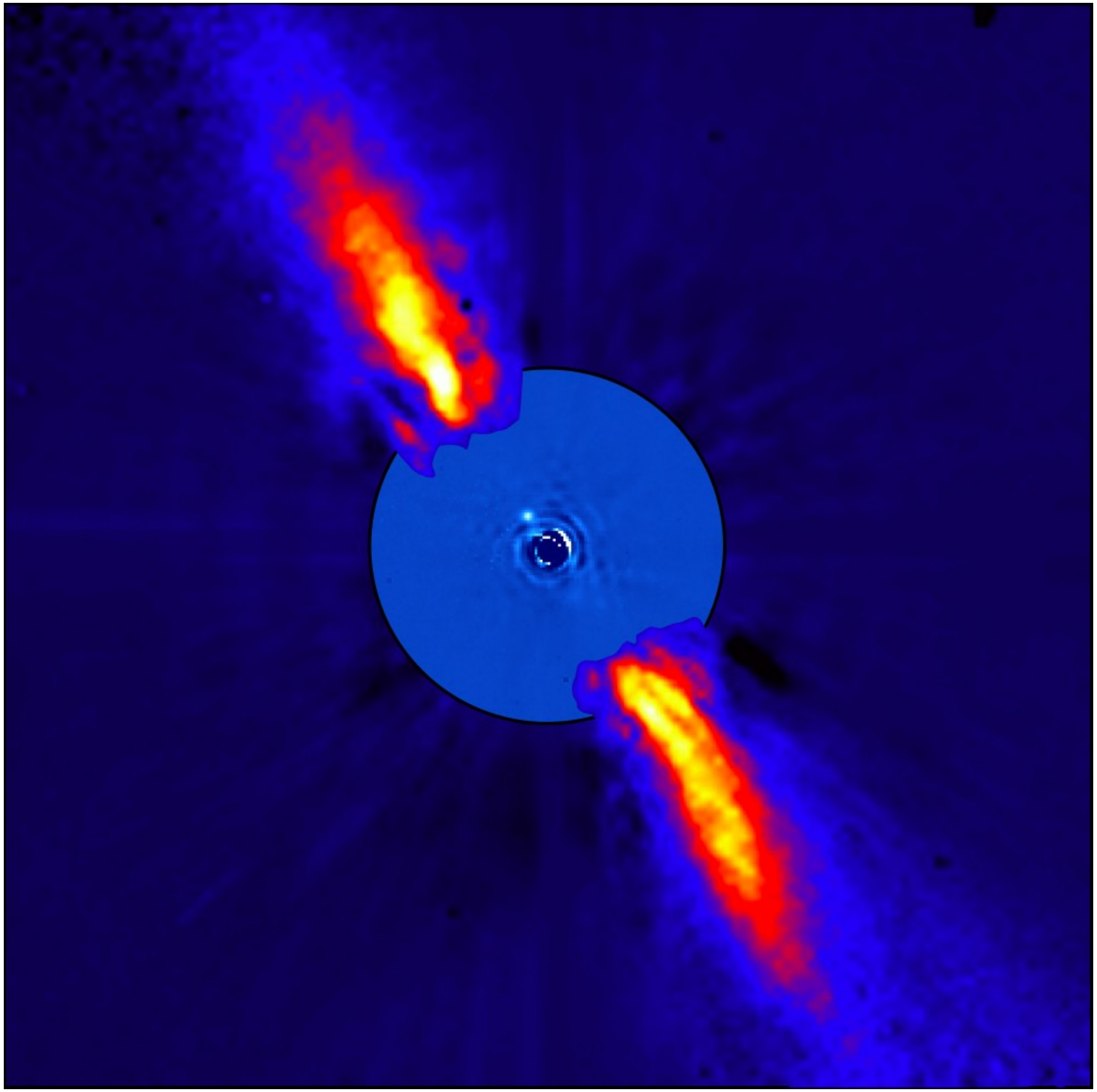
  The factorization of infrared divergences and the dipole formula

- Two projects:

I. The breaking of the Regge factorization

II. Webs and possible corrections to the dipole formula

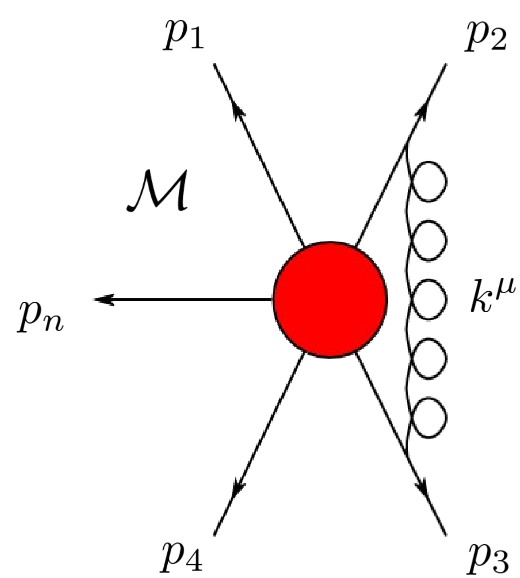
Infrared divergences



Soft and collinear gluons

- Consider a fixed angle scattering amplitude with n massless external (coloured) particles:

$$p_i \cdot p_j = O(Q^2) \gg \Lambda^2_{QCD} \;\Rightarrow \text{Hard scale}$$



- Two regions of the phase space are dangerous:

$$\boxed{\begin{array}{c} k^\mu = O(\lambda\sqrt{Q^2}) \\ \lambda \ll 1 \\ \forall \mu = 1\ldots 4 \end{array}}$$

$$\boxed{k^\mu \propto p_i^\mu \quad \text{e.g.} \quad \begin{array}{l} k^+ = \mathrm{O}\left(\sqrt{Q^2}\right) \\ k^- = \mathrm{O}\left(\lambda^2\sqrt{Q^2}\right) \\ k_\perp = \mathrm{O}\left(\lambda\sqrt{Q^2}\right) \end{array}}$$

In both cases, propagators are close to the mass shell!

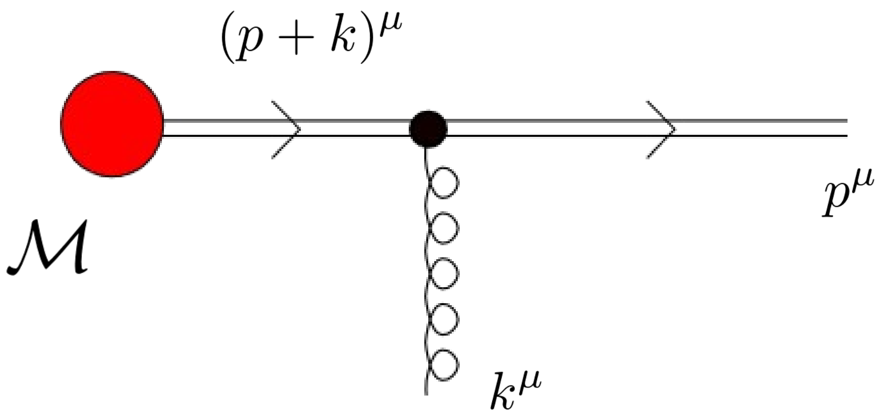
$$\boxed{\begin{array}{l} k^2 = O(\lambda^2 Q^2) \sim 0 \\ (p_i + k)^2 = 2p\cdot k = O(\lambda Q^2) \sim 0 \end{array}}$$

These configurations generate divergences which are not removed by the renormalization. At n loops the leading divergence is due to overlapping soft and collinear singularities

$$\mathcal{M} \sim \frac{1}{\epsilon^{2n}}$$

The eikonal approximation

This is not the end of the story. We evaluate the contribution of soft gluons by introducing the **eikonal approximation**:



$$\overline{u}(p)T^a\gamma^\mu \frac{\not{p}+\not{k}}{(k+p)^2}\mathcal{M} \sim \overline{u}(p)T^a\gamma^\mu \frac{\not{p}}{2k\cdot p}\mathcal{M}$$

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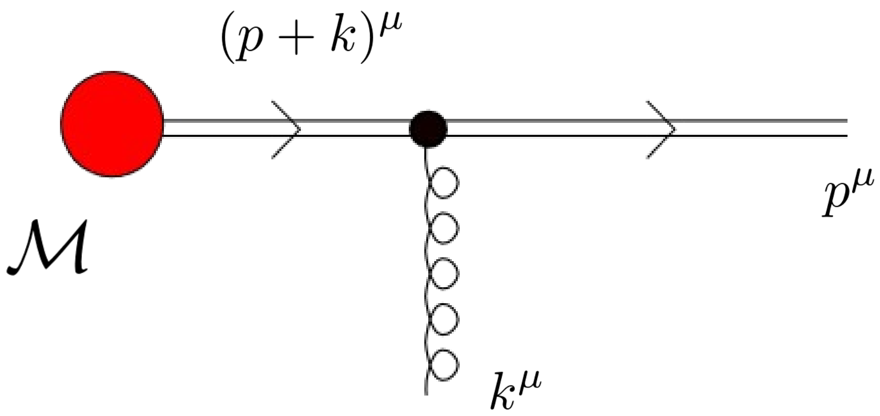

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Dirac algebra for the numerator

$$\begin{aligned} \overline{u}(p)\gamma^\mu \not{p} &= \overline{u}(p) \left[-\not{p} + 2p^\mu \right] \\ &= 2p^\mu \overline{u}(p) \end{aligned}$$

$$\overline{u}(p)\mathcal{M} \times T^a \frac{p^\mu}{k \cdot p}$$

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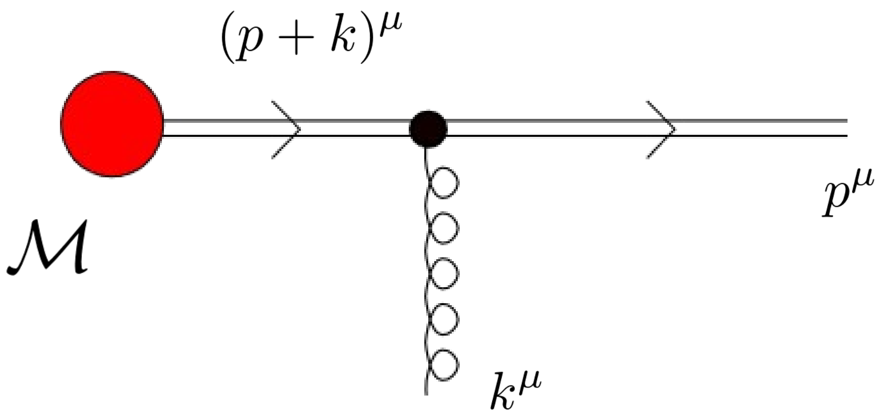

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The emission of a soft gluon is *factorized.*

We extract the *eikonal* Feynman rules.

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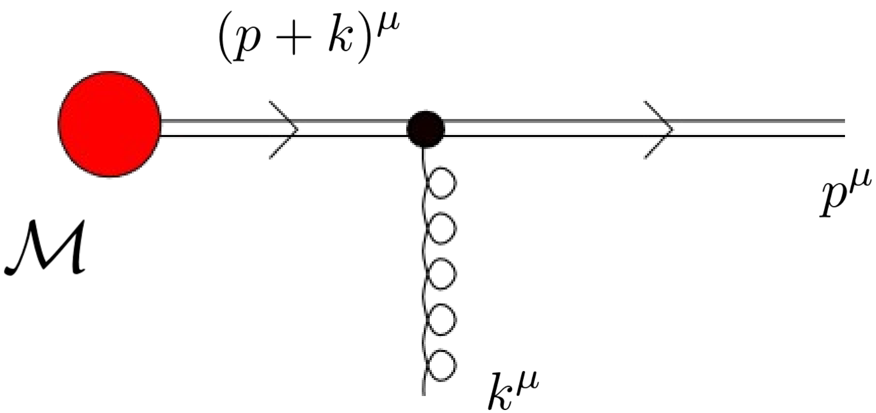

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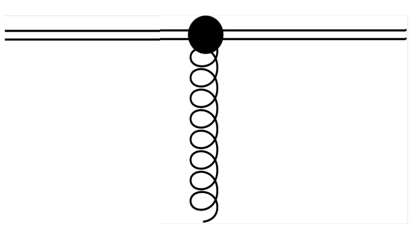


$$\text{Vertex}: g_s\, p^\mu T^a$$

$$\text{Propagator}: \frac{1}{p\cdot k}$$

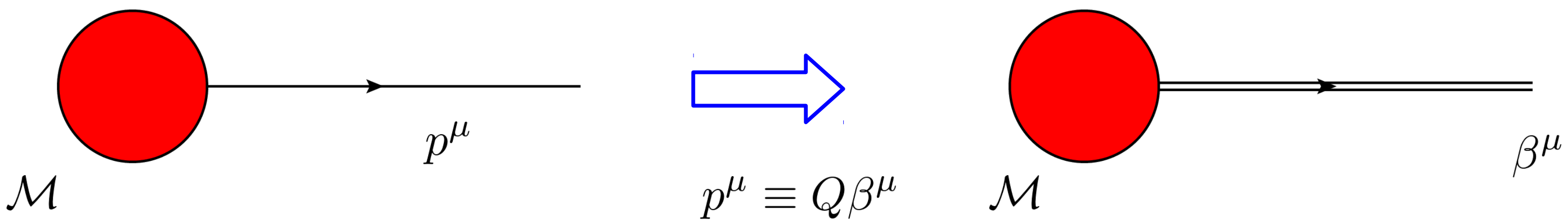
Wilson lines

Eikonal Feynman rules are

- independent on the spin of the emitting parton
- Independent on the energy of the emitting parton because of the invariance

$$\mathrm{p}^{\mu} \to \alpha p^{\mu} \qquad \frac{p^{\mu}}{p \cdot k} \quad \text{invariant}$$

The same Feynman rules are obtained by replacing the hard parton by a semi-infinite Wilson line:



The Wilson line is the path ordered exponential of the gauge field integrated along the classical trajectory of the hard parton:

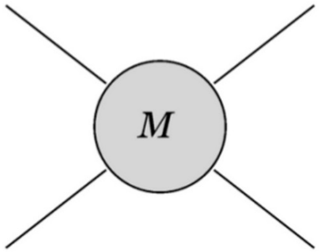
$$\mathcal{W}_{\beta}(\infty, 0) = \mathcal{P}exp\left[ig_s \int_0^{\infty} dx \; A^{\mu}(x\beta) \cdot \beta_{\mu}\right]$$

The Wilson line depends only on:

- The velocity of the hard parton, through the path of integration
- The colour charge of the hard parton, through the representation of the field A^{μ}

Soft collinear factorization

The divergences of scattering amplitudes are factorized and they are universal



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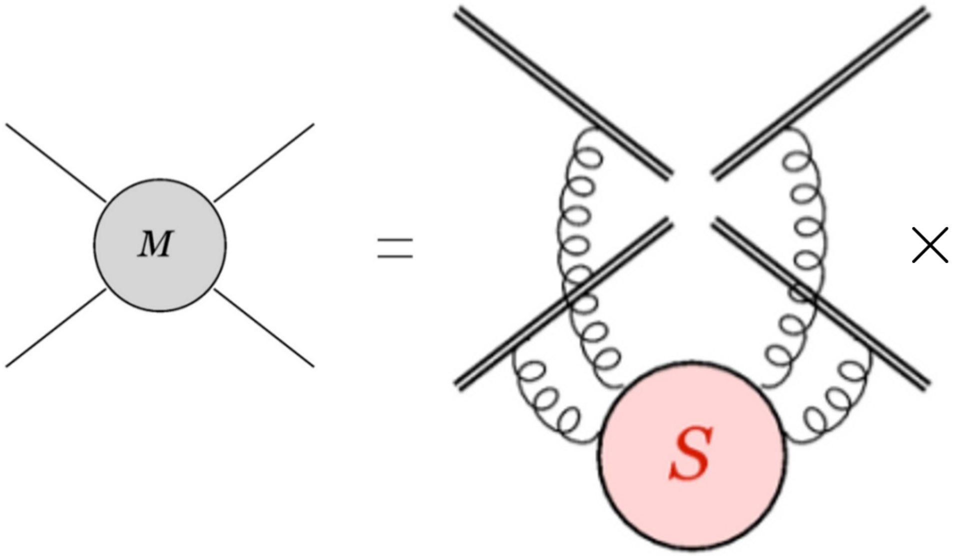

The soft singularities are generated by the process independent operator S, correlator of Wilson lines.

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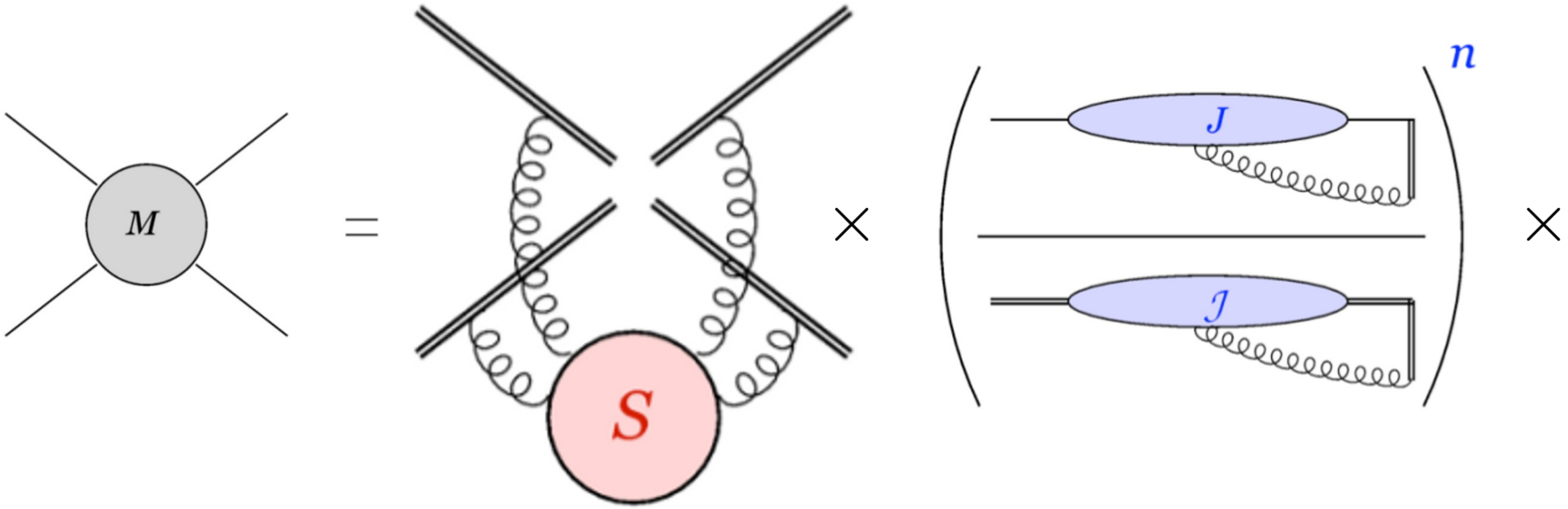

The soft singularities are generated by the process independent operator S, correlator of Wilson lines.

Collinear singularities are generated by a product of jet functions, each depending on one external parton.

The eikonal jet in the denominator subtracts the overlapping soft and collinear singularities, to avoid double counting.

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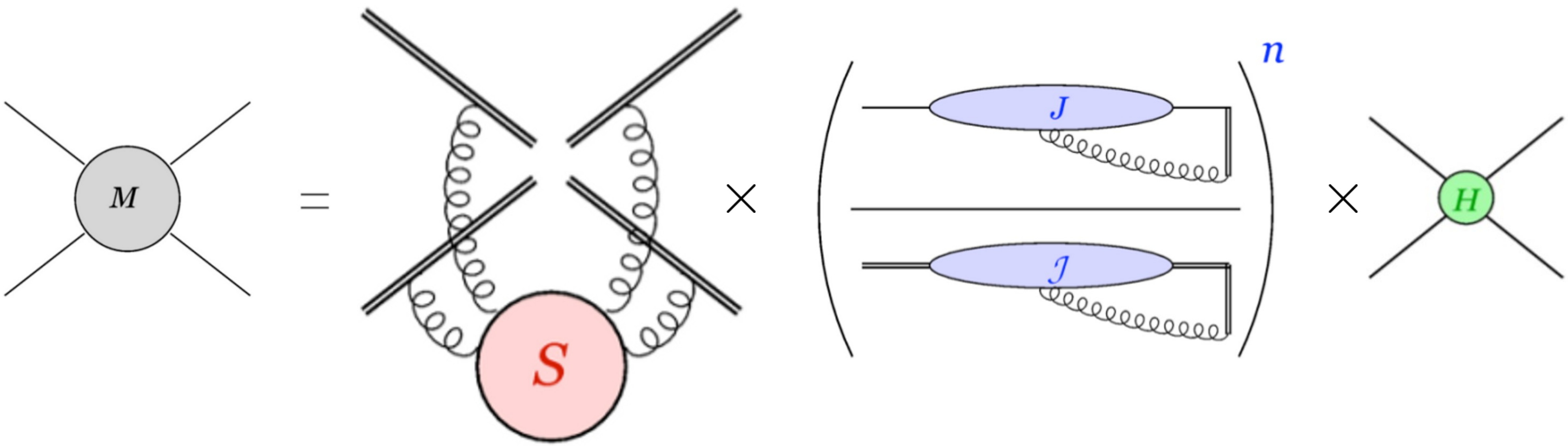

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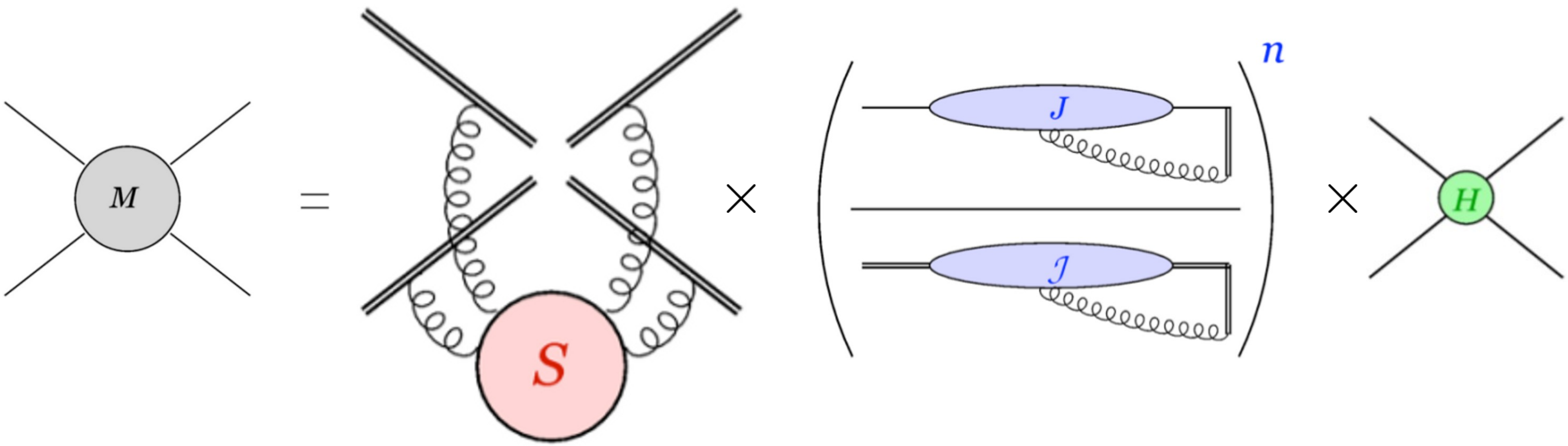

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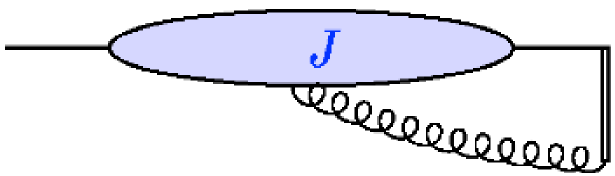
Finite, process dependent hard function

$$\mathcal{M}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = Z\left(\frac{p_i}{\mu_F}, \alpha_s(\mu_F^2), \epsilon\right)\,\mathcal{H}\left(\frac{p_i}{\mu}, \frac{\mu_F}{\mu}, \alpha_s(\mu^2)\right)$$

Definitions

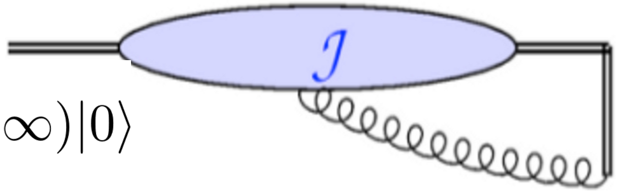
Gauge invariant definitions of the soft and jet functions make use of Wilson lines and their correlators

Jet function:



$$J_i\left(\frac{(p_i\cdot n_i)^2}{\mu_F^2},\alpha_s(\mu_F^2)\right)=\langle 0|W_{n_i}(\infty,0)\psi(0)|p\rangle$$

Eikonal jet function:

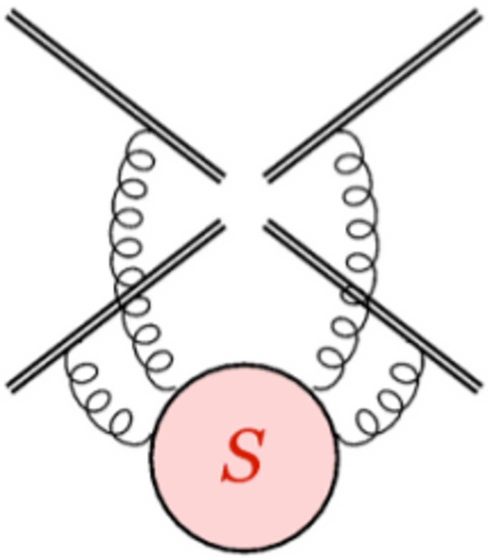


$$\mathcal{J}_i\left(\frac{(\beta_i\cdot n_i)^2}{n_i^2},\alpha_s(\mu_F^2)\right)=\langle 0|W_{n_i}(\infty,0)W_{\beta_i}(0,-\infty)|0\rangle$$

Soft function:

$$S\left(\beta_i\cdot\beta_j,\alpha_s(\mu_F)\right)=\langle 0|\mathcal{W}_{\beta_1}\ldots\mathcal{W}_{\beta_n}|0\rangle$$

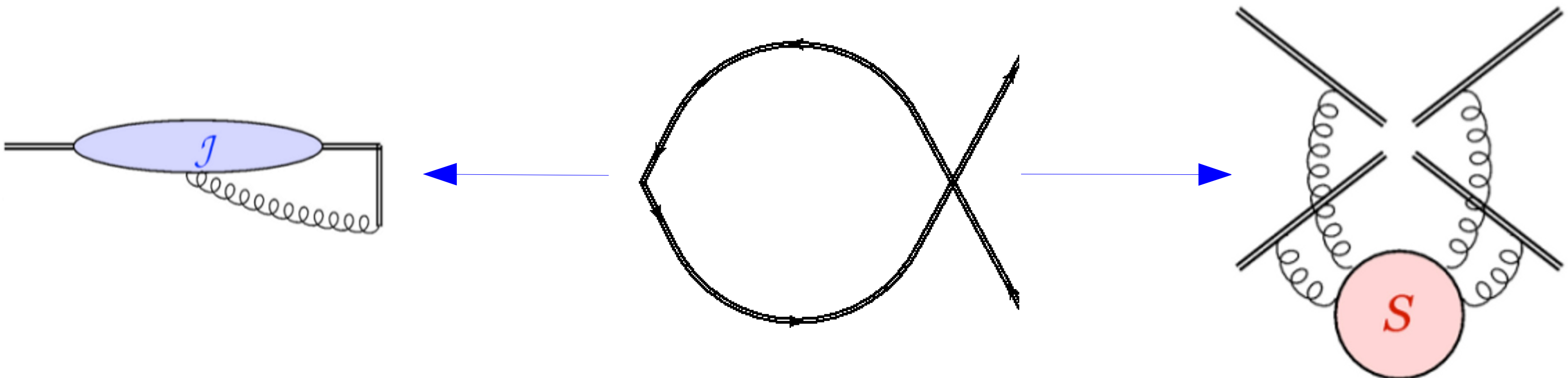
$$Z\left(\frac{p_i}{\mu_F},\alpha_s(\mu_F)\right)=S\times\prod_n\left(\frac{J_i}{\mathcal{J}_E}\right)^n$$



Evolution equations

The divergences of the soft and the jet functions are related to the renormalization proprieties of Wilson loops: indeed

- The eikonal jets are cusped Wilson loops
- The soft function can be interpreted as a self-intersecting Wilson loop



$$\mu_F \frac{d}{d\mu_F} \mathcal{J}_E \left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu_F^2), \epsilon \right) = -\gamma_E \, \mathcal{J}_E \left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu_F^2), \epsilon \right)$$

$$\mu_F \frac{d}{d\mu_F} S \left(\beta_i \cdot \beta_j, \alpha_s(\mu_F^2), \epsilon \right) = -S \left(\beta_i \cdot \beta_j, \alpha_s(\mu_F^2), \epsilon \right) \times \Gamma_S \left(\beta_i \cdot \beta_j, \alpha_s(\mu_F^2), \epsilon \right)$$

The partonic jet satifies a renormalization group equation too

$$\mu_F \frac{d}{d\mu_F} J_i \left(\frac{p_i}{\mu_F}, \alpha_s(\mu_F^2) \right) = -\gamma_i \, J_i \left(\frac{p_i}{\mu_F}, \alpha_s(\mu_F^2) \right)$$

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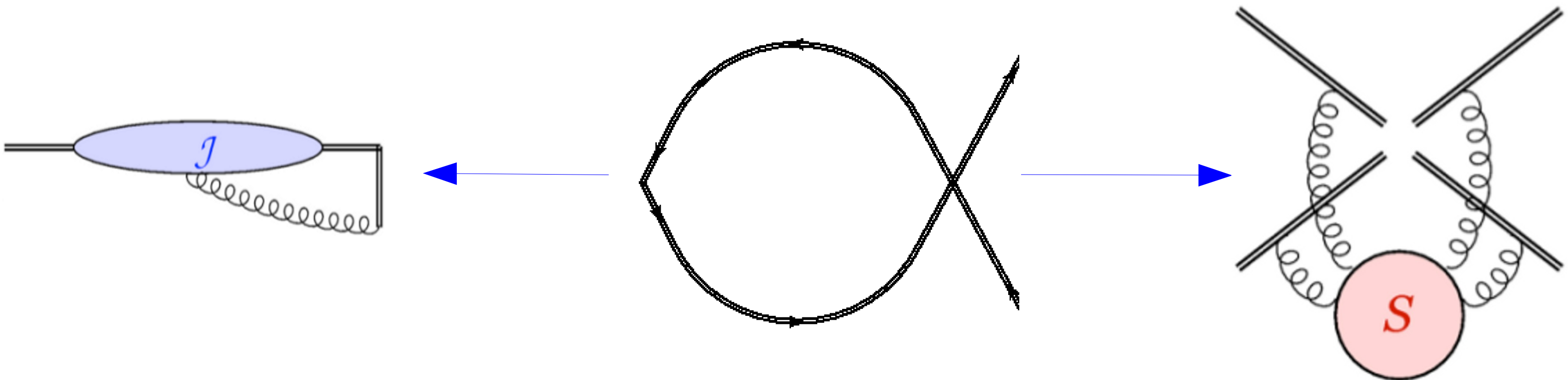

$$\mu_F \frac{d}{d\mu_F} \mathcal{J}_E \left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu_F^2), \epsilon \right) = -\gamma_E \, \mathcal{J}_E \left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu_F^2), \epsilon \right)$$

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Finally

$$\boxed{\mu_F \frac{d}{d\mu_F} Z \left(\frac{p_i}{\mu_F}, \alpha_s(\mu_F^2), \epsilon \right) = -Z \left(\frac{p_i}{\mu_F}, \alpha_s(\mu_F^2), \epsilon \right) \Gamma \left(\frac{p_i}{\mu_F}, \alpha_s(\mu_F^2) \right)}$$

The dipole ansatz

- The anomalous dimension Γ is the key ingredient to find the singularities of the amplitude

$$Z\left(\frac{p_i}{\mu_F}, \alpha_s(\mu_F), \epsilon\right) = \mathcal{P}exp\left[-\int_0^{\mu_F} \frac{d\mu}{\mu}\Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu)\right)\right]$$

- The dipole formula is an all orders ansatz for Γ, correct up to two loops

$$\Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = -\frac{1}{4}\widehat{\gamma}_K \sum_{i=1}^{n}\sum_{j\neq i}\log\left(\frac{2|p_i\cdot p_j|e^{-i\pi\lambda_{ij}}}{\mu^2}\right)\boxed{T_i\cdot T_j} + \sum_{i=1}^{n}\gamma_i\,\boxed{1}$$

$$\widehat{\gamma}_K(\alpha_s) = \frac{\gamma_K(\alpha_s)}{C_i} = 2\frac{\alpha_s(\mu^2)}{\pi} + \left[\left(\frac{67}{18}-\frac{\pi^2}{6}\right)C_A - \frac{5}{9}n_f\right]\frac{\alpha_s^2(\mu^2)}{\pi^2}$$

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$$\gamma_i(\alpha_s) = \gamma_i^{(1)}\frac{\alpha_s(\mu^2)}{\pi}+\gamma_i^{(2)}\frac{\alpha_s^2(\mu^2)}{\pi^2}$$

- Divergences arise when we integrate the running coupling around $\mu\to 0$

$$K(\alpha_s(\mu^2),\epsilon)\equiv -\frac{1}{4}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\widehat{\gamma}_K(\alpha_s(\lambda^2,\epsilon)) = \frac{\alpha_s}{\pi}K^{(1)}+\frac{\alpha_s^2}{\pi^2}K^{(2)}+\ldots \qquad K^{(1)}=\frac{1}{2\epsilon}$$

Similarly we define

$$D(\alpha_s(\mu^2),\epsilon)\equiv -\frac{1}{4}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\widehat{\gamma}_K(\alpha_s(\lambda^2,\epsilon))\log\left(\frac{\mu^2}{\lambda^2}\right) \qquad B_i(\alpha_s(\mu^2),\epsilon)\equiv -\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\gamma_i(\alpha_s(\lambda^2,\epsilon))$$

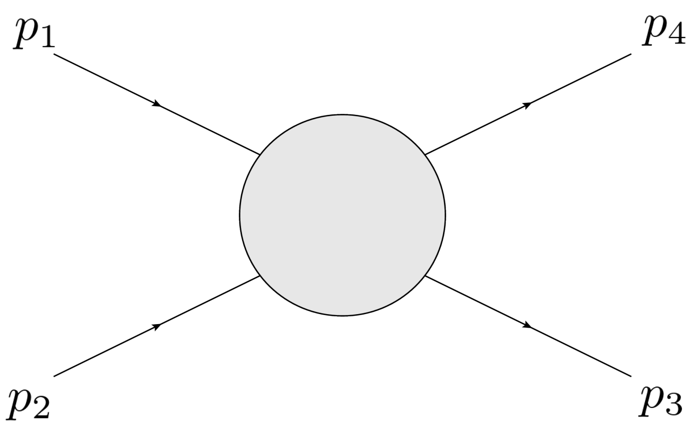
Scattering amplitudes

- $2 \to 2$ scattering amplitude with massless partons

$$\boxed{\begin{aligned} s &= (p_1 + p_2)^2 \\ t &= (p_1 - p_4)^2 \\ u &= (p_1 - p_3)^2 \end{aligned}}$$

Kinematics

$$s + t + u = 0$$



$$\boxed{\begin{aligned} T_s^2 &= (T_1 + T_2)^2 \\ T_t^2 &= (T_1 + T_4)^2 \\ T_u^2 &= (T_1 + T_3)^2 \end{aligned}}$$

Colour

$$T_s^2 + T_t^2 + T_u^2 = \sum_{i=1}^{4} C_i$$

$$\mathcal{M} = Z \times \mathcal{H}$$

Finite but process dependent: we have to determine for each scattering amplitude

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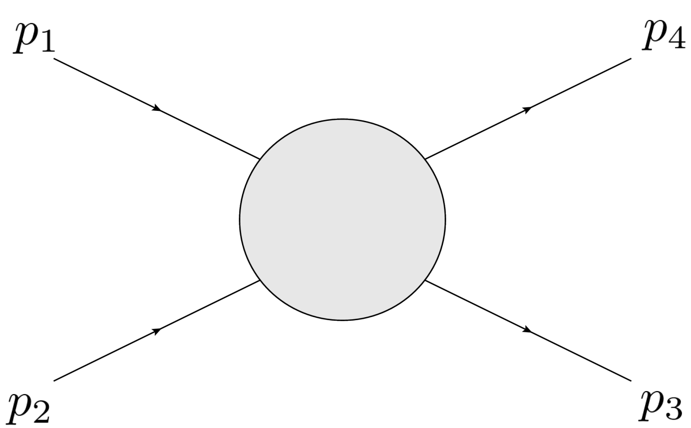

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$$\mathcal{M} = Z \times \mathcal{H}$$

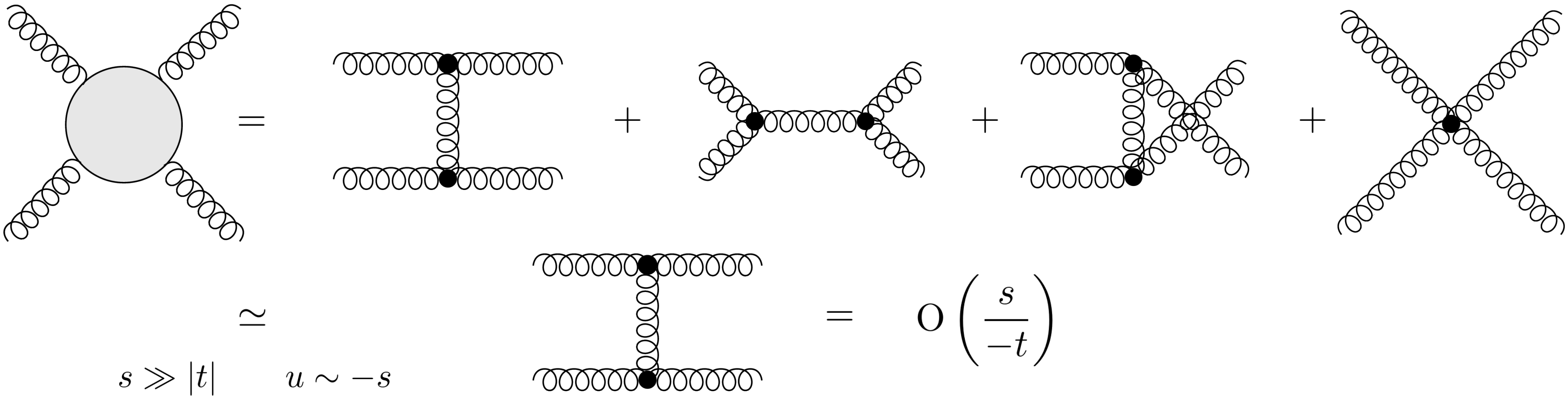
Finite but process dependent: we have to determine for each scattering amplitude

$$Z = exp\left\{\int_0^{\mu^2}\frac{d\lambda}{\lambda}\left[\frac{1}{4}\widehat{\gamma}_K\left[\boxed{\log\left(\frac{se^{-i\pi}}{\lambda^2}\right)}\left(\boxed{T_s^2}-\frac{1}{2}\sum_{i=1}^{4}C_i\right)\right.\right.\right.$$
$$+\boxed{\log\left(\frac{-t}{\mu^2}\right)}\left(\boxed{T_t^2}-\frac{1}{2}\sum_{i=1}^{4}C_i\right)+\boxed{\log\left(\frac{-u}{\mu^2}\right)}\left(\boxed{T_u^2}-\frac{1}{2}\sum_{i=1}^{4}C_i\right)\Bigg]$$
$$\left.\left.-\frac{1}{2}\sum_{i=1}^{4}\gamma_{J_i}(\alpha_s(\lambda^2))\right]\right\}$$

Reggeization and its breaking

The high energy limit

Scattering amplitudes in the high energy limit are dominated by a subset of diagrams: consider the gluon gluon scattering in a physical gauge



The exchange of a gluon in the t channel is the leading diagram, the other terms are power suppressed. At higher orders, the amplitude is dominated by the exchange of a ladder in the t channel and it has large logarithms:

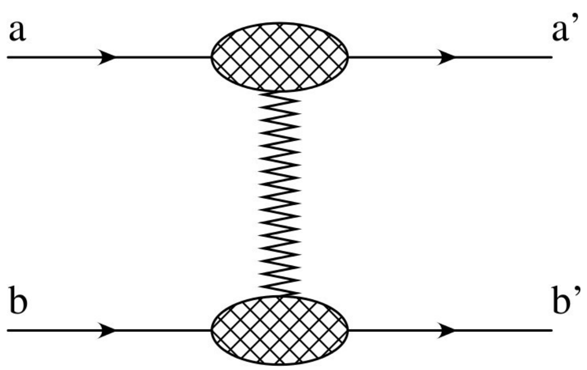
$$\mathcal{M}^{n\mathrm{loops}} \simeq \boxed{\log^{n}\left(\frac{s}{-t}\right) M^{(n),n}} + \boxed{\log^{n-1}\left(\frac{s}{-t}\right) M^{(n),n-1}} + \ldots$$

Leading logarithms

Next-to-Leading logarithms

The Regge factorization

The Regge formula resums all the leading logarithms of the amplitude by dressing the gluon propagator at all orders:



$$\mathcal{M} = g_s^2 \frac{s}{t} \boxed{T^{\sigma}_{a'a} T^{\sigma}_{b'b}} \times \boxed{C_{aa'} C_{bb'}} \left(\frac{s}{-t} \right)^{\boxed{\alpha(-t)}}$$

T-channel diagonal colour structure

Effective vertices

The Regge trajectory governs the large logarithms

$$\alpha(-t) = \frac{\alpha_s}{\pi} \alpha^{(1)} + \frac{\alpha_s^2}{\pi^2} \alpha^{(2)} + \ldots$$

$$C_{ij} = C_{ij}^{(0)} \left(1 + \frac{\alpha_s}{\pi} C_{ij}^{(1)} + \frac{\alpha_s^2}{\pi^2} C_{ij}^{(2)} + \ldots \right)$$

The accuracy of the Regge formula is extended to the next-to-leading logarithms by writing it manifestly symmetric under

$$s \leftrightarrow u \sim -s$$

consistently with the exchange of ladders in the t channel

$$\mathcal{M} = g_s^2 \frac{s}{t} T^{\sigma}_{a'a} T^{\sigma}_{b'b} \times C_{aa'} C_{bb'} \left\{ \left[\left(\frac{s}{-t} \right)^{\alpha(-t)} + \left(\frac{-s}{-t} \right)^{\alpha(-t)} \right] + \kappa \left[\left(\frac{s}{-t} \right)^{\alpha(-t)} - \left(\frac{-s}{-t} \right)^{\alpha(-t)} \right] \right\}$$

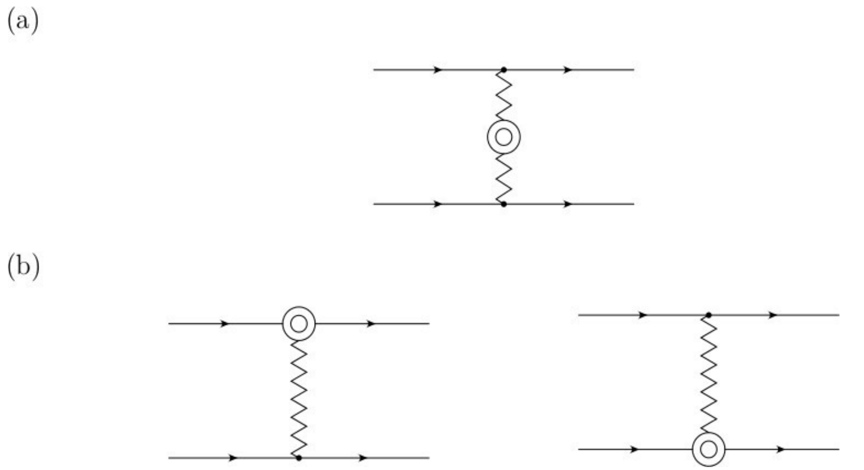
$\kappa = 0$ gg or qg scattering

$\kappa = \frac{4-N^2}{N^2}$ qq scattering

Universality of the Regge formula

The perturbative expansion of the Regge formula can be compared to the fixed order calculations: at one loop we have

$$\mathcal{M}^{1\text{ loop}} = \left[\alpha^{(1)} \log\left(\frac{s}{-t}\right) + C_{ii}^{(1)} + C_{jj}^{(1)}\right]\mathcal{M}^{(0)}$$



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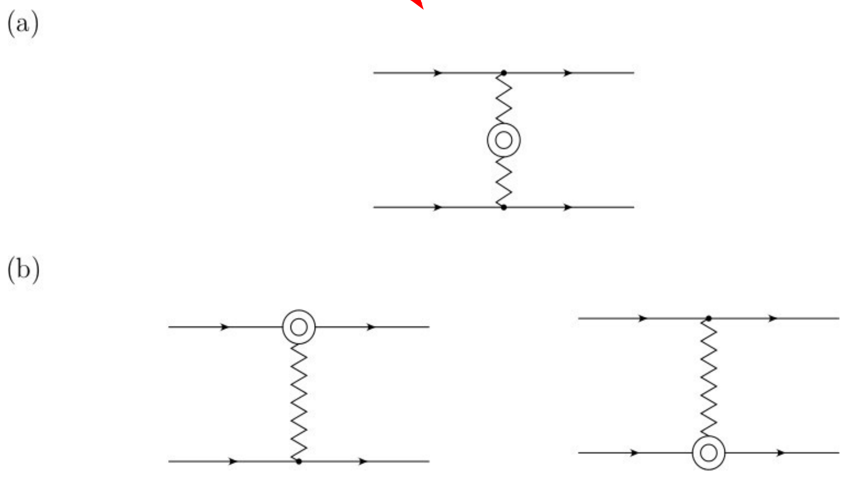

Universality of the Regge formula

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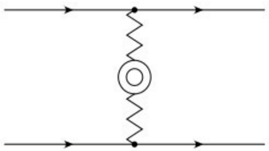
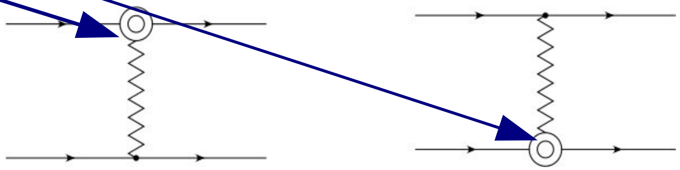


(b)



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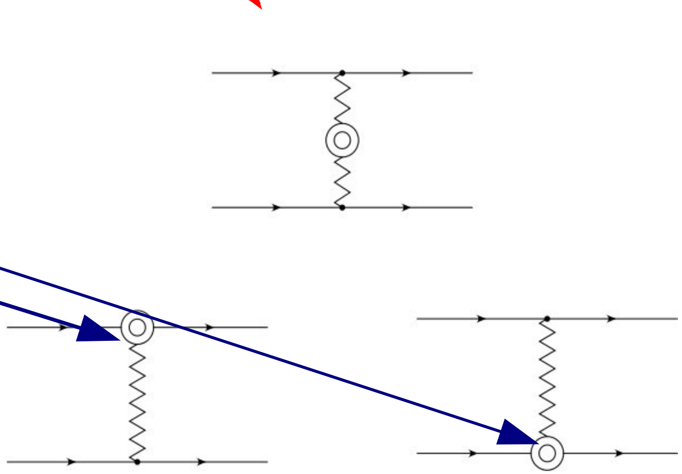
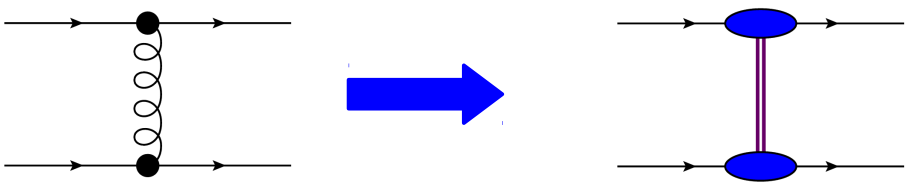


We can extract these coefficients from the explicit amplitudes:



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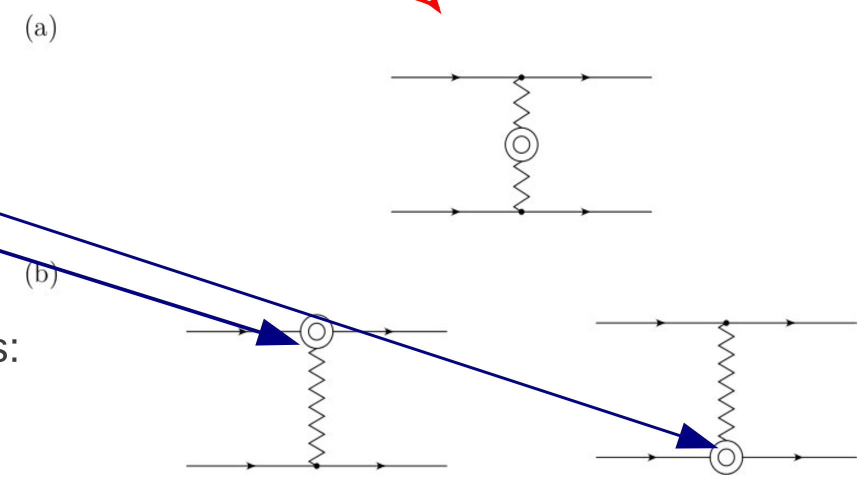

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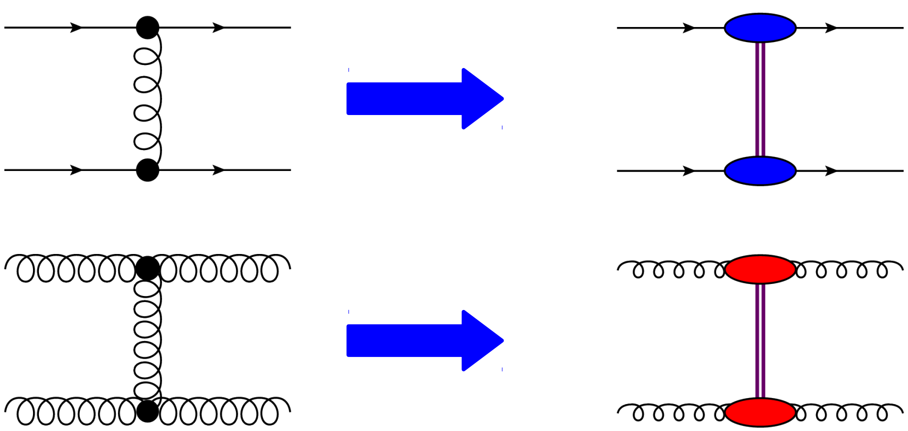

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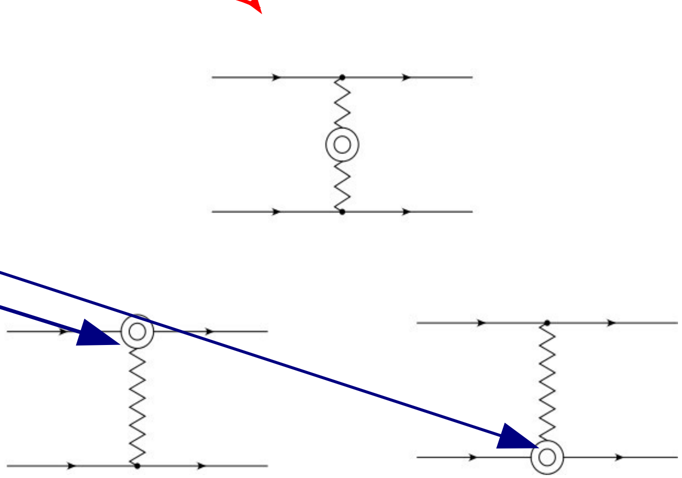

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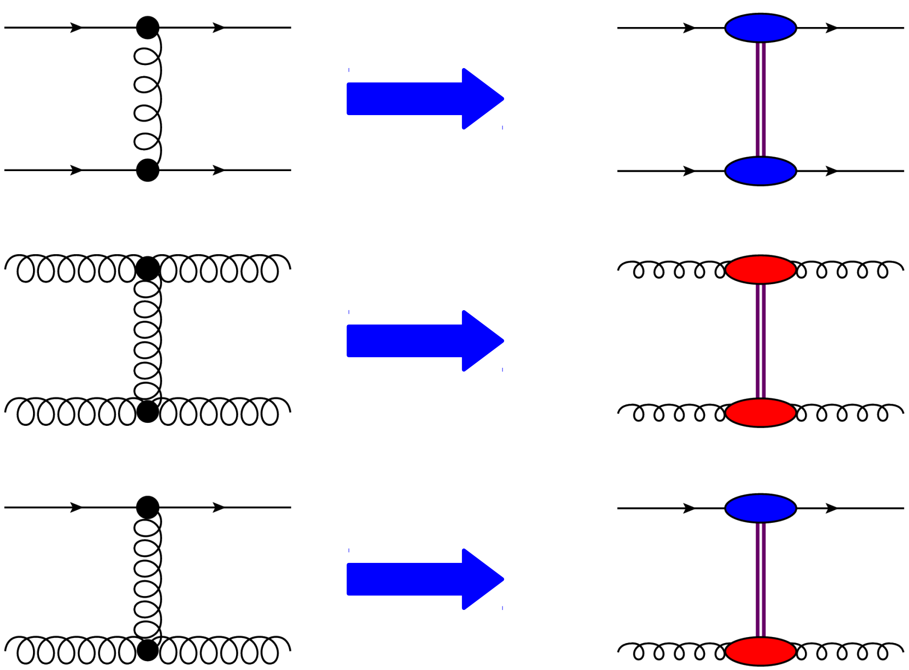


The quark-gluon scattering amplitude can be used as a consistency check.

$$\frac{\mathcal{M}_{qg}^{(1)}}{\mathcal{M}_{qg}^{(0)}} = \alpha^{(1)} \log\left(\frac{s}{-t}\right) + C_{qq}^{(1)} + C_{gg}^{(1)}$$

Breaking of the Regge factorization

LL and NLL are correctly predicted by the Regge formula

$$\mathrm{Re}\left(\mathcal{M}_{qg}\right) = \left\{ \frac{(\alpha^{(1)})^2}{2} \boxed{\log\left(\frac{s}{-t}\right)^2} \right.$$

$$\left[\alpha^{(2)} + \alpha^{(1)}\left(C_q^{(1)} + C_g^{(1)}\right)\right] \boxed{\log\left(\frac{s}{-t}\right)}$$

$$\left. \boxed{+C_q^{(2)} + C_g^{(2)} + C_q^{(1)} C_g^{(1)} - \frac{\pi^2}{2}(1+\kappa)(\alpha^{(1)})^2} \right\} \mathcal{M}_{\mathrm{tree}}$$

The NNLL fails at the level of the double pole (Del Duca, Glover):

$$\mathrm{Re}(\mathcal{M}_{qg}) - \left[C_q^{(2)} + C_g^{(2)} + C_q^{(1)} C_g^{(1)} - \frac{\pi^2}{2}(1+\kappa)(\alpha^{(1)})^2\right] \mathcal{M}_{\mathrm{tree}} = \frac{3\pi^2}{16\epsilon^2}\left(\frac{N^2+1}{N^2}\right)$$

We want to use the information about the infrared structure in the dipole formula to explain the difference.

Dipole formula at high energy

- We consider the high energy limit and drop the power suppressed terms

$$s \gg -t \Rightarrow \begin{array}{l} \log\left(\frac{-u}{\mu^2}\right) = \log\left(\frac{s+t}{\mu^2}\right) = \log\left(\frac{s}{\mu^2}\right) + O\left(\frac{-t}{s}\right) \\ T_u^2 = \sum_{i=1}^4 \mathcal{C}_i - T_s^2 - T_t^2 \end{array}$$

- The dipole formula in the high energy limit is

$$Z = \exp\left\{\boxed{K(\alpha_s(\mu^2))\left[T_t^2 \log\left(\frac{s}{-t}\right) + i\pi T_s^2\right]}\right.$$
$$\left. + \underbrace{\boxed{\sum_i B_i(\alpha_s(\mu^2)) + \frac{1}{2}\sum_i \mathcal{C}_i \left[D(\alpha_s(\mu^2)) + K(\alpha_s(\mu^2)) \log\left(\frac{-t}{\mu^2}\right)\right]}}_{\equiv Z_{\mathbf{1}}} - \frac{i\pi}{2} K(\alpha_s(\mu^2)) \sum_i \mathcal{C}_i \right\}$$

$\equiv Z_{\mathbf{1}}$ Colour singlet

- The operator Z has large logarithms: in the Leading Log approximation we keep only the dominant contributions.

$$\mathcal{M}_{LL} = \left(\frac{s}{-t}\right)^{K(\alpha_s(\mu^2))T_t^2} Z_{\mathbf{1}} \mathcal{H} = \left(\frac{s}{-t}\right)^{\boxed{K(\alpha_s(\mu^2))\, N}} \boxed{Z_{\mathbf{1}}}\, \boxed{\mathcal{H}}$$

The dipole formula predicts a t-channel diagonal colour structure for all the scattering processes. We expect the breaking of this behaviour in the subleading terms, because of $i\pi T_s^2$

Looking for reggeization breakings

- Why does the Regge factorization break down?

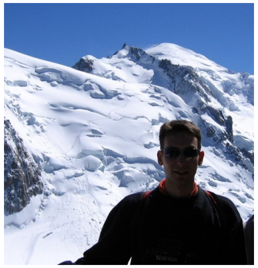
  If the reggeization breaking found in the amplitude at two loops has an infrared origin, we can use the dipole formula to better understand this point and predict the deviation from the Regge formula at three loops.

This project has been realized in collaboration with

Vittorio Del Duca (INFN Frascati e La Sapienza)

Lorenzo Magnea

Leonardo Vernazza



The results have been reported in

High-energy QCD amplitudes at two loops and beyond,

arXiv: 1311.0304

in publication on Physics Letters B.

Regge factorization vs dipoles

The idea is to expand the dipole formula at fixed order in the strong coupling and compare it with the expansion of the Regge formula.

We already noted that the dipole formula is consistent with the reggeization of the leading logarithms (and the NLL for the real part), but we can extract more information.

The comparison is simple: for example at one loop

$$\mathrm{Re}\mathcal{M}^{(1)}_{\mathrm{dipoles}} = \left(K_1(\alpha_s) N_c \mathcal{M}^{(0)} + H^{(1),1}\right) \log\left(\frac{s}{-t}\right) + Z_{1,ij}\mathcal{M}^{(0)} + \mathrm{Re}H^{(1),0}_{ij}$$

$$\mathrm{Re}\mathcal{M}^{(1)}_{\mathrm{Regge}} = \left[\alpha^{(1)} \log\left(\frac{s}{-t}\right) + C^{(1)}_{ii} + C^{(1)}_{jj}\right]\mathcal{M}^{(0)}$$

We get the impact factors and Regge trajectory at one loop:

$$\alpha^{(1)} = K_1(\alpha_s) N_c + \text{finite parts}$$

$$C^{(1)}_{ii} = \frac{1}{2} Z^{(1)}_{1,ii} + \text{finite parts}$$

Soft gluons effects at NNLL

If we go beyond the LL in the dipole formula we have contributions of the T_s^2

$$Z = \exp\left[K(\alpha_s)\left(\log\left(\frac{s}{-t}\right)T_t^2 + i\pi T_s^2\right)\right] Z_1 \Rightarrow \begin{array}{l} -\dfrac{\pi^2 K(\alpha_s)^2}{2}(T_s^2)^2\ Z_1 \\ -\frac{\pi^2}{\epsilon^3}K^3(\alpha_s)\log\left(\frac{s}{-t}\right)\left[T_s^2,\left[T_t^2,T_s^2\right]\right] Z_1 \end{array}$$

In the expansion up to two loops and in the real part we find a non diagonal colour structure in the t-channel.

We compare again the dipole formula and the Regge formula to find the impact factors

Factorized contribution depending only on parton i

$$C_{ii}^{(2)} = \boxed{\frac{Z_{1,ii}^{(2)}}{2} - \frac{(Z_{1,ii}^{(1)})^2}{8} + \frac{1}{4}Z_{1,ii}^{(1)}\frac{\mathrm{Re}H^{(1),0}}{\mathcal{M}^{(0)}}}$$
$$\boxed{-\frac{\pi^2(K_1^{(1)})^2}{4}\left[\left(T_{s,ij}^2\right)^2_{[8],[8]} - \mathcal{C}_{tot}(T_{s,ij}^2)_{[8],[8]} + \frac{\mathcal{C}_{tot}^2}{4} - \frac{1+\kappa}{4}N^2\right]} + \mathrm{O}\left(\epsilon^0\right)$$

Non universal dependence on the other external partons

Non factorizing rests

We generalize the Regge formula by introducing a non factorizing rest, which depends on the scattering process.

$$\mathcal{M} = g_s^2 \frac{s}{t} T^{\sigma}_{a'a} T^{\sigma}_{b'b} \Bigg\{ C_{aa'} C_{bb'} \left[\left(\frac{s}{-t} \right)^{\alpha(-t)} + \left(\frac{-s}{-t} \right)^{\alpha(-t)} \right] + \kappa\, C_{aa'} C_{bb'} \left[\left(\frac{s}{-t} \right)^{\alpha(-t)} - \left(\frac{-s}{-t} \right)^{\alpha(-t)} \right] + R_{ab}(\alpha) \Bigg\}$$

We can introduce the non factorizing part of the impact factor in the rest at two loops

$$\boxed{R^{(2),0}_{ij} = -\pi^2 (K_1^{(1)})^2 \left[(T^2_{s,ij})^2_{[8],[8]} - \mathcal{C}_{tot} (T^2_{s,ij})_{[8],[8]} + \frac{\mathcal{C}^2_{tot}}{4} - \frac{1+\kappa}{4} N^2 \right]}$$

$$R^{(2),0}_{qq} = \frac{\pi^2}{4\epsilon^2} \left(1 + \frac{3}{N^2} \right) \qquad R^{(2),0}_{gg} = -\frac{3\pi^2}{2\epsilon^2} \qquad R^{(2),0}_{qg} = -\frac{\pi^2}{4\epsilon^2}$$

We recover

$$\boxed{\begin{aligned} \mathrm{Re}(\mathcal{M}_{qg}) - \left[C_q^{(2)} + C_g^{(2)} + C_q^{(1)} C_g^{(1)} - \frac{\pi^2}{2} (1+\kappa) (\alpha^{(1)})^2 \right] \mathcal{M}^{(0)} &= \frac{1}{2} \left[R^{(2),0}_{qg} - \left(\frac{R^{(2),0}_{qq} + R^{(2),0}_{gg}}{2} \right) \right] \\ &= \frac{3\pi^2}{16\epsilon^2} \left(\frac{N^2+1}{N^2} \right) \end{aligned}}$$

Three loops expansion

The ambiguity in the definition of the impact factors at two loops breaks the Regge factorization at three loops at the level of the single logs:

$$\mathcal{M}_{ij}^{(3),1} = \Big[\alpha^{(3)} + \alpha^{(2)}\left(C_i^{(1)} + C_j^{(1)}\right) + \alpha^{(1)}\left(C_i^{(2)} + C_j^{(2)} + C_i^{(1)}C_j^{(1)}\right) - \frac{(\alpha^{(1)})^3}{4}(1+\kappa)\Big]\mathcal{M}^{(0)}$$

By comparing the dipole formula expansion we identify the non-factorizing rest at three loops

$$\boxed{\alpha^{(3)}} + \boxed{\frac{R^{(3),1}}{2}} = \boxed{K^{(3)}(\alpha_s)N} + \boxed{\frac{\pi^2 (K^{(1)})^3}{2}\Big[\mathcal{C}_{tot} N (T^2_{s,ij})_{[8],[8]} - \frac{\mathcal{C}^2_{tot} N}{4}} \boxed{+ \frac{1+\kappa}{2}N^3 - \frac{1}{3}\sum_n (2N + \mathcal{C}_n)\left|(T^2_{s,ij})_{[8],n}\right|^2\Big]} + \mathrm{O}\left(\epsilon^{-1}\right)$$

$$R_{qq}^{(3),1} = \frac{\pi^2}{\epsilon^3}\frac{2N^2-5}{12N} \qquad R_{gg}^{(3),1} = -\frac{\pi^2}{\epsilon^3}\frac{2N}{3} \qquad R_{qg}^{(3),1} = -\frac{\pi^2}{\epsilon^3}\frac{N}{24}$$

Prediction on the amplitude at three loops

Summary and outlook

- We found the origin of the breaking of Regge factorization in QCD scattering amplitudes at two loops, by expanding the dipole formula. We have a prediction on the leading reggeization breaking effects at three loops.

Next....

- We are using the Regge factorization up to NLL to derive constraints on the hard parts, which are out of control in the dipole formula.

- We are going to write down explicitly all the poles predicted by the dipole formula in the amplitudes at three loops in the high energy limit, by introducing extra information of the amplitudes at two loops.

- We will study the quark reggeization in the quark gluon backward scattering in order to find the quark Regge trajectory and impact factors up to two loops and their singularities at three loops.

Another paper is in preparation

Corrections to the dipole formula

Beyond the dipole formula

- There are two possible sources of corrections to the dipole formula.
  At three loops, the anomalous dimension can include:

The correction begins with a correlation of four partons, present at least at three loops.

$$\Gamma = \Gamma_{\text{dipole}} + \Delta\left(\frac{\rho_{ij}\rho_{kl}}{\rho_{ik}\rho_{jl}}\right)$$

$$\rho_{ij} = \frac{(\beta_i \cdot \beta_j)^2}{\frac{(\beta_i \cdot n_i)^2}{n_i^2}\frac{(\beta_j \cdot n_j)^2}{n_j^2}}$$

- At four loops, the Casimir scaling of the cusp anomalous dimension can break down:

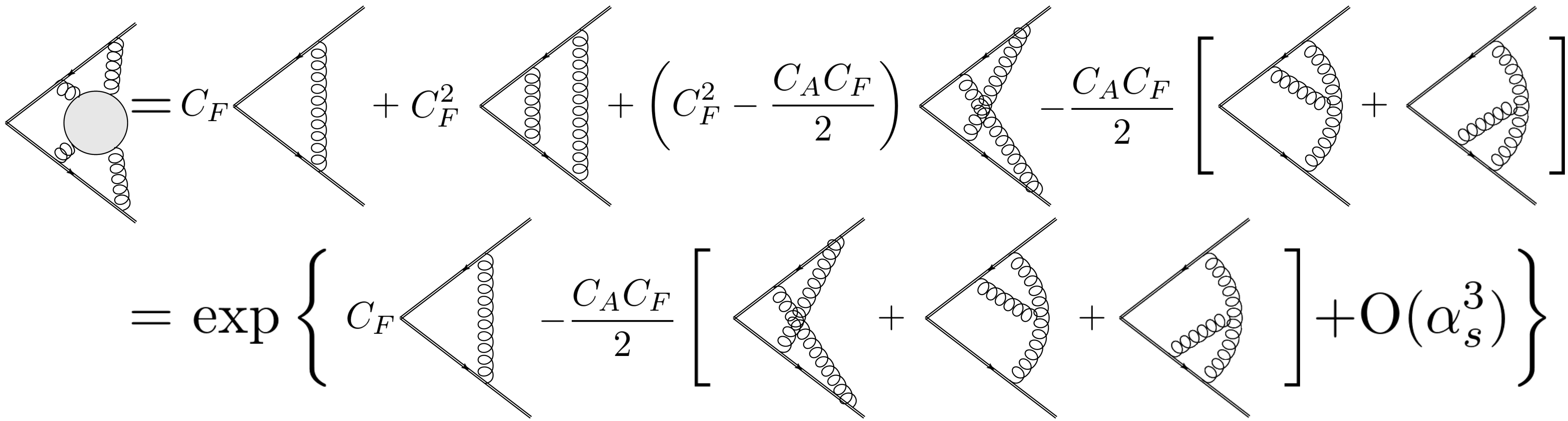
$$\gamma_K \neq \mathcal{C}_i \widehat{\gamma}_K(\alpha_s)$$
$$\gamma_K \to \mathcal{C}_i \widehat{\gamma}_K(\alpha_s) + \tilde{\gamma}_K(\alpha_s)$$

The correction can begin with the quartic Casimir.

We focus on the first type of corrections and compute the soft anomalous dimension at three loops.

Computational tool: webs (I)

The soft anomalous dimension is computed as a correlator of Wilson lines. These object are calculated by exponentiating a subset of all the possible Feynman diagrams with the appropriate *exponentiated colour factors.* The simplest expample is with only two lines at two loops:



The *antisymmetric* combination of the diagrams which differ for exchange of two gluons on the Wilson line is multiplied by a modified colour factor. The *symmetric* combination doesn't appear in the expenent. In a formal way we can write:

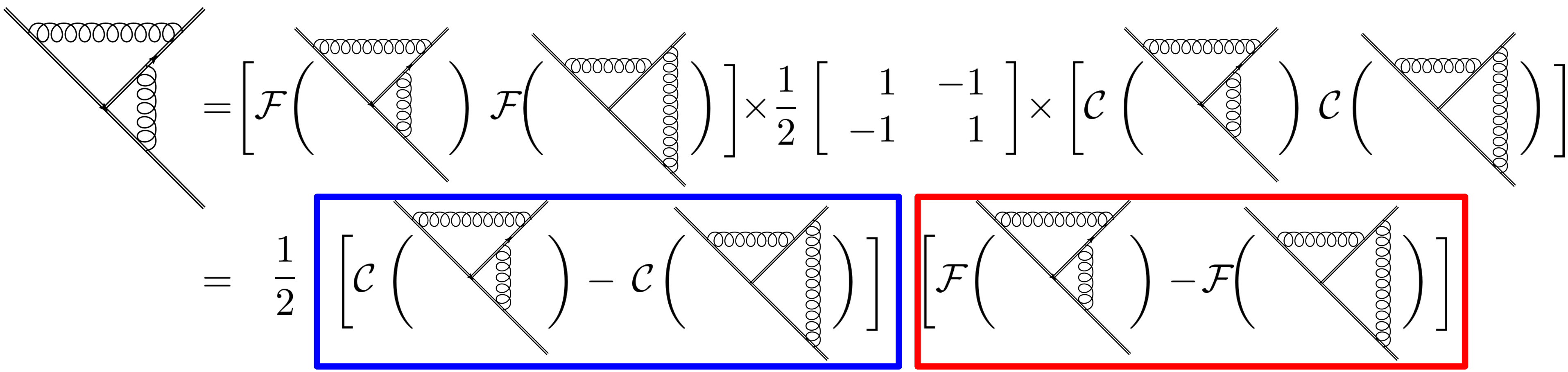
$$\langle 0 | W_{\beta_1}(\infty, 0) W_{\beta_2}(\infty, 0) | 0 \rangle = \exp \sum_{DD'} \mathcal{F}(D) \, R_{DD'} \mathcal{C}(D'))$$

$$R_{DD'} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$\mathcal{F}(D)$	is the kinematic factor of diagram D
$\mathcal{C}(D')$	is the colour factor of diagram D'
$R_{DD'}$	is the web mixing matrix

Webs (II)

In the multileg case the same formula holds and there is a combinatoric algorithm to compute the mixing matrix.



Colour structure of the web

Kinematic part

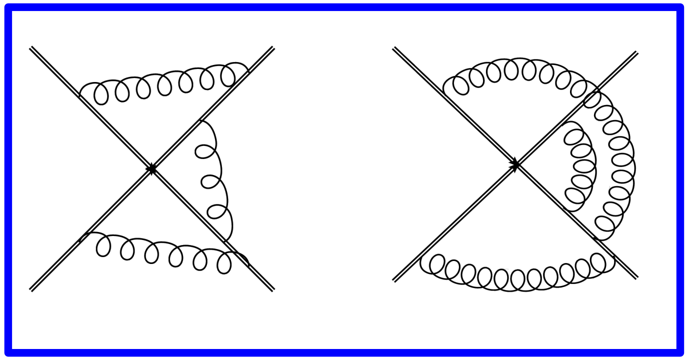
We obtain the soft function by exponentiating the webs $$S(\beta_i \cdot \beta_j, \alpha_s, \epsilon) = \exp\left[\sum_n w_n\right]$$

We compute the anomalous dimension by differentiating the soft function. In terms of webs we need to include the commutators of lower orders because of the expansion of the matrix exponential.

$$\mu \frac{d}{d\mu} S \equiv -S \times \Gamma \rightarrow \begin{array}{l} \Gamma^{(1)} = -2w^{(1,-1)} \\ \Gamma^{(2)} = -4w^{(2,-1)} - 2\left[w^{(1,-1)}, w^{(1,0)}\right] \end{array}$$

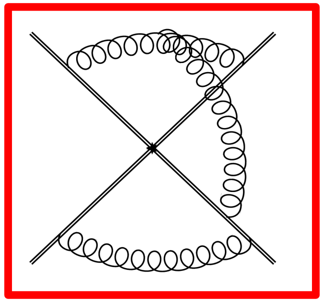
Four partons correlations

The correlations of four partons at three loops are given by the diagrams



These object have been recently calculated: the key features are

- Factorized, dipole-like kinematic dependence on the cusp angles.
- Both webs are written in terms of the basis of functions:

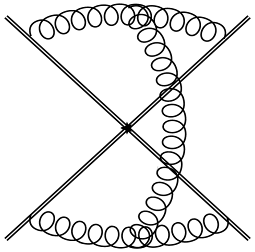


$$M_{k,n}(x,\gamma_{ij}) = \int_0^1 dx\, p_0(x,\gamma_{ij}) \log^k\left(\frac{q(x,\gamma_{ij})}{x^2}\right) \log^{2n}\left(\frac{x}{1-x}\right)$$

$$\text{with} \quad p_0(x,\gamma_{ij}) = \frac{1}{x^2+(1-x)^2-\gamma x(1-x)}$$

$$\log q(x,\gamma) = -\log p_0(x,\gamma_{ij})$$

$$\gamma_{ij} = 2\frac{\beta_i \cdot \beta_j}{\sqrt{\beta_i^2 \beta_j^2}}$$

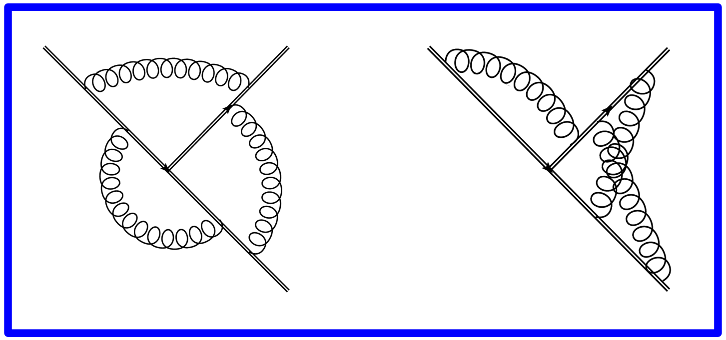


At this perturbative order, the results contain

$$M_{2,0} \quad \text{and} \quad M_{0,1}$$

Work on the anomalous dimension

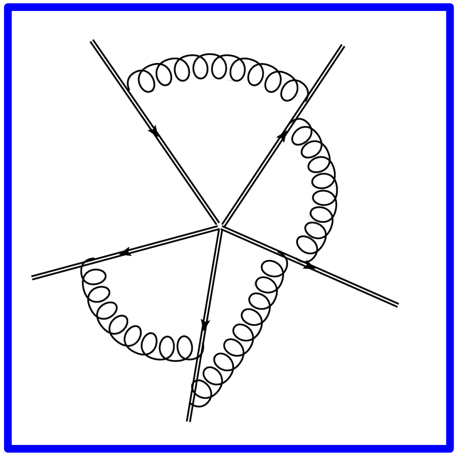
- Is there a basis of functions describing all the webs at three loops involving only single gluon exchanges? Look more entangled situations



The kinematic dependence is still factorized: the first diagram is written in terms of the already known functions, the second needs the new function

$$\overline{\Sigma}_1 = \int_0^1 dx\, dy\, \theta(x-y) p_0(x,\gamma_{ij}) p_0(y,\gamma_{ij}) \log\left(\frac{x}{1-x}\right)$$

- Does the kinematic factorization hold at higher loops?



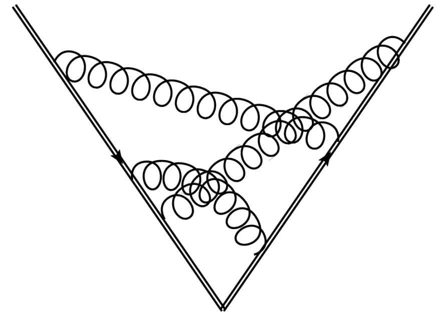
Yes! The result is expressed in terms of the same basis. At this order we find

$$M_{3,0}(\gamma_{ij}),\ M_{1,1}(\gamma_{ij})$$

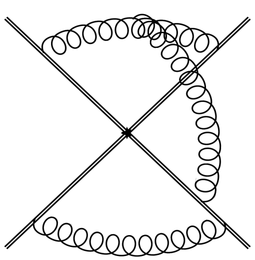
Summary and outlook

A paper in preparation about:

- the study of the basis of functions for the anomalous dimension at three loops, including the fully entangled web correlating only two partons
- an ansatz of the basis of functions for all multiloop single gluon exchange webs
- the first four loop calculation



We are also proceeding in the search for a correlation of four partons at three loops in the web



Both projects are developed in collaboration with

Einan Gardi, Mark Harley (Edinburgh)
Chris White (Glasgow)
Lorenzo Magnea

Thank you!