Rigid string contributions to the interquark potential from the lattice
Second year Seminar

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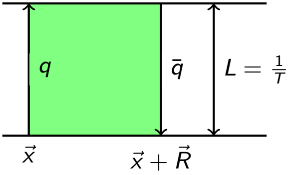
Introduction
Physical setup

Polyakov loop:

$$P(\vec{x}) = Tr\, P\, e^{-\int_0^{1/T} \mathrm{d}x_0 A_0(x_0,\vec{x})}$$

order parameter for confinement in pure gauge theories



$$\langle P(\vec{x})\rangle \begin{cases} 0 & T \leq T_c \\ \neq 0 & T > T_c \end{cases}$$

For two static ($M_q, M_{\bar{q}} \to \infty$) charges

$$\langle P^\dagger(\vec{x}) P(\vec{x})\rangle = e^{-LV(R)}$$

Confinement : $V(R) \to_{R\to\infty} \sigma R$, σ string tension

- For some unknown reason, the dominant configurations in the presence of distant charges are those in which the flux lines form a thin tube.
- We describe the dynamics of the flux tube (at $T \sim 0$) in order to describe the low energy regime of the theory in the confining phase.

Introduction

Effective String Theory

Why study an effective theory of strings?

- Many other examples of stable string-like objects in field theory.
- Strong constraints from symmetry: effective string theories much more predictive than other effective theories in particle physics (universal behaviour).

In the case of the confining string: deviations from the universal behaviour could be a signature of the true theory of the QCD string!

Assumptions:

- The field theory in D euclidean spacetime dimensions has a mass gap.
- No string breaking effects.

Effective String Theory

Spacetime symmetry and string formation

The flux tube formation breaks some of the spacetime symmetries of the underlying field theory.

Goldstone's theorem

Given a field theory with an internal symmetry group G, broken to a stability group H, there are

$$dimG - dimH$$

massless modes in that ground state. One for each broken generator.

From the breaking pattern

$$ISO(D) \to SO(D-2) \times ISO(2),$$

we expect $3(D-2)$ Goldstone modes. However, we use the $(D-2)$ transversal displacements of the string only!

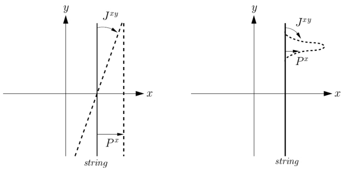
We cannot naively apply Goldstone's theorem to broken spacetime symmetries!

Effective String Theory

Spacetime symmetry and string formation

In the case of broken spacetime symmetries, the number of independent massless modes is reduced by the number n_x of vanishing linear combinations of transformations[1]

$$dimG - dimH - n_x.$$



$$c_a(r)T^a\langle\phi(r)\rangle = 0,$$

$c_a(r)$ unknown function, T^a broken generator, $\langle\phi(r)\rangle$ order parameter.

In the case of the effective string, those are the $(D-2)$ transversal displacements w.r.t its straight equilibrium position

$$X^i = X^i(\xi_0, \xi_1), \quad i = 1, \cdots, D-2..$$

[1] (Low and Manohar, 2002)

Effective String Theory

Massless worldsheet excitations

To preserve the stability group, the action must be built with contracted products of $\partial_\alpha X_i$

$$\partial_\alpha X \cdot \partial^\alpha X, \quad \partial_\alpha X \cdot \partial_\beta X \; \partial^\alpha X \cdot \partial^\beta X, \cdots$$

and is a derivative expansion

$$S_{\text{eff}} = \sigma RL + \frac{\sigma}{2} \int \mathrm{d}^2\xi \left[(\partial_\alpha X \cdot \partial^\alpha X) + c_2 \left(\partial_\alpha X \cdot \partial^\alpha X \right)^2 + c_3 \left(\partial_\alpha X \cdot \partial_\beta X \right)^2 + \cdots \right] + S_b$$

with

$$S_{\text{b}} = \int \mathrm{d}\xi_0 \left[b_1 (\partial_1 X \cdot \partial_1 X) + b_2 (\partial_1 \partial_0 X \cdot \partial_1 \partial_0 X) + b_3 (\partial_1 X \cdot \partial_1 X)^2 + \cdots \right]$$

the boundary contribution. A rescaling $\xi_0 \to \xi_0 R$, $\xi_1 \to \xi_1 L$ and $X_i \to X_i / \sqrt{\sigma}$ shows that S is actually a long string expansion in powers of $(\sigma RL)^{-1}$.

Effective String Theory

Non linear realization of symmetries

Even if the system lies in a broken ground state, Poincaré symmetry must still be present in the action. The symmetry of the remaining $2(D-2)$ generators is realized in a non-linear manner[2]. For a rotation involving the directions b and j

$$\delta^j_{\epsilon b} X_i = \epsilon \left(-\delta^j_i - X^j \partial_b X_i \right)$$

- The terms of the long string expansion mix up under this transformation.
- The mixing only involves terms with equal weight

$$W = n(derivatives) - n(fields).$$

The imposition of Poincaré symmetry

$$\delta_\epsilon S = 0$$

generates recurrence relations among the $\{c_i\}$ and the $\{b_i\}$ and the number of free parameters is greatly reduced[a]!

[a] (Gliozzi, 2011)

[2] (Meyer, 2006)

Effective String Theory

Classification of allowed terms

The allowed terms can be classified according to their weight.

- $W = 0$:

$$S_{NG} = \sigma \int d^2\xi \sqrt{h} \ , \ \ h = \det(h_{ab}) = \det\left(\partial_a X \cdot \partial_b X\right) .$$

- $W = 2$:

$$S_{2,R} = \gamma \int d^2\xi \sqrt{h}\, \mathcal{R} \ , \ \ S_{2,K} = \alpha \int \mathrm{d}^2\xi \sqrt{h}\ K^2 \ ,$$

  with $K = \Delta(h)X$ extrinsic curvature, $\mathcal{R}$ Ricci scalar.

- Both can be neglected at the classical level[3]:
  - In D=3, $\mathcal{R}$ is a topological invariant.
  - K is proportional to the weight 0 equation of motion.

- $W = 4$,

$$S_{4,R} = \gamma_2 \int d^2\xi \sqrt{h}\mathcal{R}^2 \ , \cdots$$

[3](Aharony and Field, 2011)

Effective String Theory

The reparametrization invariant approach

The same results can be obtained with a reparametrization invariant approach:

$$X^\mu : \mathcal{M} \to \mathbb{R}^D, \;\; \mu = 0, 1, \cdots, D-1$$

with $\mathcal{M}$ worldsheet of the string, parametrized by ξ_0, ξ_1, $\mathbb{R}^D$ target space of the gauge theory.

We require:

- Poincaré invariance acting on X^μ.
- Reparametrization invariance over $\mathcal{M}$.

Terms of the derivative expansion are geometrical invariants built from the natural geometrical objects we can define on $\mathcal{M}$

$$h_{ab} = \partial_a X^\mu \partial_b X_\mu, \;\; \Omega^\mu_{ac} = \nabla_c \partial_a X^\mu$$

where ∇_c is the covariant derivative induced on $\mathcal{M}$ by h_{ab}. The previous approach can be obtained by fixing the gauge

$$X_0 = \xi_0, \; X_1 = \xi_1, \;\; \text{physical gauge}$$

Effective String Theory
Summary and a new proposal

In both approaches:

- The contribution from first order terms of the action coincide with NG[4].
- The first allowed deviations from NG in $D = 3$ appear at order $O(R^{-7})$.

However, deviations much stronger than $O(R^{-7})$ are observed:

- SU(N) excited string spectrum[5].
- 3D Ising model ground state potential[6].
- Interquark potential of U(1) lattice gauge theory in 3D[7].

Our proposal

The contribution from weight 2 terms vanishes at tree level, but may contribute at 1-loop!

[4] (Aharony and Field, 2011)
[5] (Athenodorou et al., 2007)
[6] (Caselle et al., 2003)
[7] (Caselle et al., 2014)

The Rigid String

The static interquark potential

Our proposal

The contribution from weight 2 terms vanishes at tree level, but may contribute at 1-loop!

$$V(R) = \lim_{L\to\infty} \frac{1}{L} \log \int [DX] e^{-S_{\mathrm{eff}}[X]},$$

Up to order $O(1/R^4)$ in $V(R, L)$

$$S_{\mathrm{eff}} = S_{NG} + S_{2,K} + S_{\mathrm{b}}$$

with

$$S_{NG} \simeq \sigma RL + \frac{\sigma}{2} \int \mathrm{d}^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X - \frac{1}{4} \partial_\alpha X \cdot \partial^\alpha X^2 \right] \quad (1)$$

$$S_{2,K} \simeq \alpha \int \mathrm{d}^2\xi \, (\Delta X)^2 , \quad S_{\mathrm{b}} \simeq b_2 \int \mathrm{d}\xi_0 \; [\partial_1 \partial_0 X \cdot \partial_1 \partial_0 X] \quad (2)$$

At the gaussian level, neglecting the boundary contribution

$$S_{\mathrm{eff}} = \sigma RL + \frac{\sigma}{2} \int \mathrm{d}^2\xi \, (\partial_\alpha X \cdot \partial^\alpha X) + \alpha \int \mathrm{d}^2\xi \, (\Delta X)^2$$

The Rigid String
Field transformations

$$S_R = \sigma \int d^2\xi \left[1 + \frac{1}{2} X \left(1 - \frac{1}{m^2}\Delta\right)(-\Delta) X\right], \;\; m = \sqrt{\frac{\sigma}{2\alpha}}$$

The field transformation

$$X'\left(\xi_0, \xi_1\right) = \left(1 - \frac{1}{m^2}\Delta\right)^{1/2} X\left(\xi_0, \xi_1\right)$$

takes the action back to non-rigid gaussian form,

$$S_{gauss} = \sigma \int d^2\xi \left[1 - \frac{1}{2} X \Delta X\right] + O\left(\frac{\alpha}{\sigma}\right)^2,$$

and

$$V(R) = \sigma R + \frac{1}{2L} Tr \log\left(-\Delta\right)_{RxL} = \sigma R - \frac{\pi}{24R},$$

where the singular operator trace can be evaluated with the zeta function regularization method.

The Rigid String

The static interquark potential

However, we must take into account the functional determinant of the transformation

$$\int [DX]\, exp(-S_R) = \det \left(1 - \frac{1}{m^2}\Delta\right)_{RxL}^{-1/2} \int [DX']\, exp(-S_{gauss}),$$

we obtain

$$V(R) = \lim_{L\to\infty} \left[\sigma R + \underbrace{\frac{1}{2L} Tr \log(-\Delta)_{RxL}}_{V_{NG}(R)} + \underbrace{\frac{1}{2L} Tr \log\left(1 - \frac{1}{m^2}\Delta\right)_{RxL}}_{V_r(R)} \right],$$

where

$$V_{NG}(R) = -\frac{\pi}{12R}, \quad V_r(R) = -\frac{m}{2\pi}\sum_{n=1}^{\infty}\frac{K_1(2nmR)}{n}$$

where K_α are modified Bessel functions of the second kind.

The Rigid String

Properties of the rigidity contribution

$$V_r(R) = -\frac{m}{2\pi}\sum_{n=1}^{\infty}\frac{K_1(2nmR)}{n}$$

- Has a logarithmic branching point at $R = 0$.
- Square root singularities for negative values of $(mR)^2$, the first of which at $(mR)^2 = -\pi^2$ corresponding to the convergence radius of the low mR expansion

$$V_r(R) = -\frac{\pi}{24R} + \frac{m}{4} + \frac{m^2 R}{4\pi}\left(\log\frac{mR}{2\pi} + \gamma_E - \frac{1}{2}\right) + \\ + \frac{m^2 R}{2\pi}\sum_{n=1}^{\infty}\frac{\Gamma(3/2)\zeta(2n+1)}{\Gamma(n+2)\Gamma(n-1/2)}\left(\frac{mR}{\pi}\right)^{2n}$$

- Large mR behaviour,

$$V_r(R) = \sqrt{\frac{m}{16\pi R}}e^{-2mR},\ R \gg \frac{1}{m}.$$

The $U(1)$ Lattice gauge theory in 3D

Definition

On a 3D euclidean spacetime lattice Λ (spacing a),

$$S = \beta \sum_{x\in\Lambda} \sum_{1\leq\mu<\nu\leq 3} \left[1 - \cos\vartheta_{x,\mu\nu}\right], \quad \beta = \frac{1}{ae^2}, \quad \vartheta_\mu \in (-\pi,\pi],$$

with

$$\vartheta_{x,\mu\nu} = \Delta_\mu \vartheta_{x,\nu} - \Delta_\nu \vartheta_{x,\mu}.$$

Using discrete forms notation

$$Z = \prod_{c_1} \int_{-\pi}^{\pi} \mathrm{d}(\vartheta)\, e^{-\beta \sum_{c_2}(1-\cos \mathrm{d}\vartheta)}, \; c_i\text{: i simplices}$$

where ϑ is a 1-chain, $\mathrm{d}\vartheta$ is a 2-chain and we get rid of the indices.
If ($\beta \gg 1$), taking the periodicity of S in ϑ into account

$$Z = Z_{sw} Z_{top} = Z_{sw} \sum_{\{q\}} e^{-2\pi^2\beta(q,\Delta^{-1}q)}$$

where Z_{top} describes topological excitations, Z_{sw} describes spin-waves.

The $U(1)$ Lattice gauge theory in 3D

Main properties

In the semiclassical approximation[8]

$$m_0 = c_0\sqrt{8\pi^2\beta}e^{-\pi^2\beta v(0)}, \quad \sigma \geq \frac{c_\sigma}{\sqrt{2\pi^2\beta}}e^{-\pi^2\beta v(0)}, \quad v(0) = 0.2527$$

- The model is always in the confined phase in $3D$, the bounds are saturated and semiclassically
$$c_\sigma = 8, \quad c_0 = 1$$
- The ratio
$$\frac{m_0}{\sqrt{\sigma}} = \frac{2\pi c_0}{\sqrt{c_\sigma}}(2\pi\beta)^{3/4}e^{-\pi^2 v(0)\beta/2},$$
can be tuned at will by an appropriate choice of β, in contrast to the general Yang-Mills case.

[8](Göpfert and Mack, 1981, Polyakov, 1977)

The $U(1)$ Lattice gauge theory in 3D

The dual formulation of the model[9]

Each plaquette factor in Z is periodic in ϑ

$$e^{-\beta(1-\cos \mathrm{d}\vartheta)} = \sum_{k=-\infty}^{\infty} e^{-\beta} \mathrm{I}_{|k|}(\beta) e^{\imath k \, \mathrm{d}\vartheta}, \quad I_\alpha \text{ Bessel functions of order } \alpha$$

- integration in $\mathrm{d}(\vartheta)$ yelds the constraint on each lattice plaquette

$$\delta k = 0$$

- The constraint is easily solved by defining an integer valued $^\star l$ on the dual lattice such that

$$^\star k = \mathrm{d}^\star l,$$

We obtain a globally $\mathbb{Z}$ symmetric spin model

$$Z = e^{-\beta N_l} \sum_{\{^\star l=-\infty\}}^{\{\infty\}} \prod_{^\star c_1} \mathrm{I}_{|\,\mathrm{d}^\star l|}(\beta), \quad {}^\star c_1 \text{ dual links.}$$

[9] (Savit, 1980)

The $U(1)$ Lattice gauge theory in 3D

The dual model - Insight in the confinement mechanism and Gauge/String duality

- Condensation of magnetic monopoles (pointlike in 3D, monopole rings in 4D) drives confinement!
- A string theory should describe the behaviour of Faraday flux lines connecting the sources, but this gauge/string duality is missing in the general case.

In the U(1) LGT, however, an heuristic proof exists[10]

$$S_{\mathrm{Pol}} = c_1 e^2 m_0 \int d^2\xi \sqrt{g} \; + c_2 \frac{e^2}{m_0} \int d^2\xi \sqrt{g} K^2$$

where c_1 and c_2 are two undetermined constants. If $c_1 = \sigma$ and $c_2 = 2\alpha$ then

$$\sqrt{\sigma/2\alpha} = m \sim m_0 \ .$$

and the rigidity correction is dominant in the continuum limit.

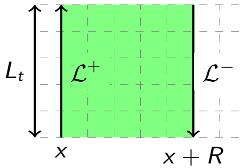
[10] (Polyakov, 1997)

The $U(1)$ Lattice gauge theory in 3D

Inclusion of Polyakov lines in the partition function

In the dual formulation, sources of the gauge field are easily included in Z

$$Z_R = \prod_{c_1} \int_{-\pi}^{\pi} \mathrm{d}\vartheta \, e^{-\beta \sum_{c_2} (1-\cos \mathrm{d}\vartheta)} \prod_{\mathcal{L}^+} e^{\imath\vartheta} \prod_{\mathcal{L}^-} e^{-\imath\vartheta}$$



$$Z_R = e^{-\beta N_l} \sum_{\{{}^\star l=-\infty\}}^{\{\infty\}} \prod_{{}^\star c_1} \mathrm{I}_{|\,\mathrm{d}{}^\star l + {}^\star n|}(\beta)$$

where ${}^\star n$ is an integer valued dual 1-chain which is nonvanishing only on the links dual to the green surface.

Thus

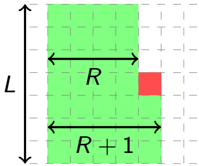
$$G(R) = \langle P^\star(R) P(0) \rangle = \frac{Z_R}{Z} (= e^{-LV(R)}),$$

Overlap problem: exponentially decaying signal-to-noise ratio!

Numericals

Evaluation of the interquark potential

Using the snake algorithm[11]...



$$\frac{G(R+1)}{G(R)} = \frac{Z_{R+1}}{Z_R} = \frac{Z_{R+1}}{Z_R^{N_t-1}} \frac{Z_R^{N_t-1}}{Z_R^{N_t-2}} \cdots \frac{Z_R^1}{Z_R}$$

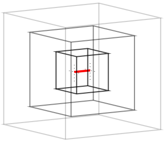
where

$$\frac{Z_R^{N_t-i+1}}{Z_R^{N_t-i}} = \left\langle \frac{\mathrm{I}_{|\,\mathrm{d}*l+1|}(\beta)}{\mathrm{I}_{|\,\mathrm{d}*l|}(\beta)} \right\rangle_{R,N_t-i}$$

are N_t independent local observables.

...and hierarchical lattice updates..

Whole lattice sweeps are a waste of computer time: we perform hierarchical lattice updates.



We obtain high precision numerical estimates of

$$\frac{G(R+1)}{G(R)} = e^{-L(V(R+1)-V(R))}$$

[11](de Forcrand et al., 2001)

Deviations from NG

Computational setup

We measured the quantity

$$Q(R) = V(R+1) - V(R) = -\frac{1}{N_t}\log\left(\frac{G(R+1)}{G(R)}\right)$$

in the range $1/\sqrt{\sigma} < R < N_t/2$ for several values of β, on lattices $L^2 x N_t$ ranging from $L = N_t = 64$ to $L = N_t = 128$.

Deviations from NG

Preliminary results

The data was fitted asymptotically with

$$Q_{NG}(R) = \sigma\left(\sqrt{(R+1)^2 - \frac{\pi}{12\sigma}} - \sqrt{R^2 - \frac{\pi}{12\sigma}}\right) ,$$

using σ as free parameter.

β	σa^2	L, N_t	$1/\sqrt{\sigma}$	$R_{\min}$
1.7	0.122764(2)	64	$3a$	$17a$
1.9	0.066824(6)	64	$4a$	$11a$
2.0	0.049364(2)	64	$5a$	$20a$
2.2	0.027322(2)	64	$6a$	$26a$
2.4	0.015456(7)	128	$8a$	$34a$

- At low β, NG describes the data for a wide range of interquark distances
- As β grows, the deviation from the prediction of NG grows: at $\beta = 2.2$ only 6 degrees of freedom!

Deviations should be detectable in the range $[a/\sqrt{\sigma}, R_{\min} a]$

Deviations with respect to NG

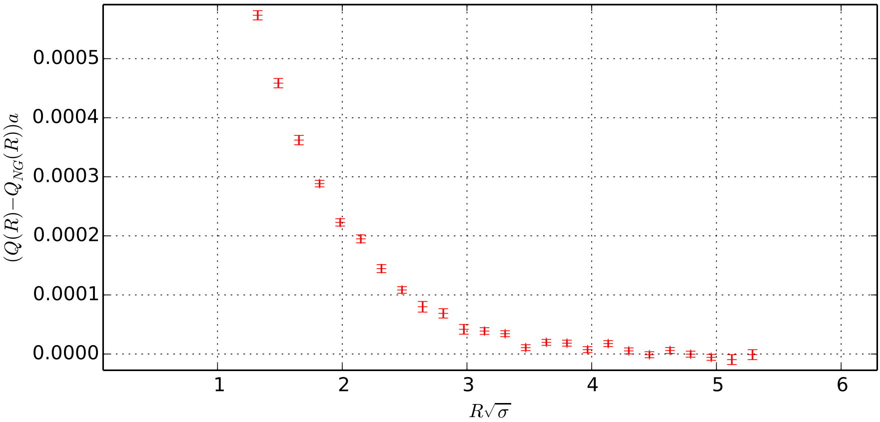


Figure: $\beta = 2.2$, $L = N_t = 64a$, $\sigma a^2 = 0.027322(2)$

Deviations with respect to NG

Boundary and rigidity terms

The boundary correction

$$Q_b(R) = -\frac{b_2\pi^3}{60}\left(\frac{1}{(R+1)^4} - \frac{1}{R^4}\right)$$

doesn't describe the deviations:

- $\chi^2_R \sim 1$ only for very large values of $R_{\min}\sqrt{\sigma}$.
- The best fit values of b_2 have the wrong scaling behaviour!

Fitting with the rigidity correction

$$Q_r(R) = -\frac{m}{2\pi}\sum_{n=1}^{\infty}\frac{K_1(2nm(R+1)) - K_1(2nmR)}{n}, \quad m = \sqrt{\frac{\sigma}{2\alpha}}$$

or at next to leading order

$$Q_r'(R) = Q_r(R) + \frac{21}{20m\sigma}\left(\frac{\pi}{24}\right)^2\left(\frac{1}{(R+1)^4} - \frac{1}{R^4}\right)$$

works much better!

Determination of *ma*

At NLO...

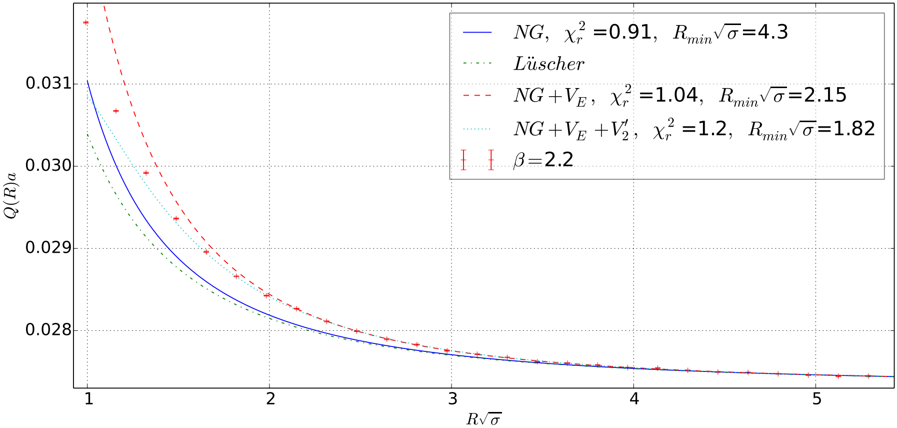
Fits of $Q(R)$ with

$$Q(R) = Q_{NG}(R) + Q_r(R) + Q_r'(R) + Q_b(R)$$

using σ, m and b_2 as free parameters:

β	ma	$m_0 a$	m/m_0
1.7	0.28(9)	0.88(1)	0.32(10)
1.9	0.25(4)	0.56(1)	0.45(7)
2.0	0.17(2)	0.44(1)	0.39(4)
2.2	0.11(1)	0.27(1)	0.41(4)
2.4	0.06(2)	0.20(1)	0.30(10)

- Takes into account the interplay between σ, m, and b_2 in the error.
- m scales with m_0 as predicted by Polyakov.



Keeping into account the above analysis, our estimate of the rigidity parameter is

$$m/m_0 = 0.35(10) \ .$$

Conclusions and Future directions

- At 1-loop, the rigidity term cannot be neglected and is in fact essential in order to explain numerical data.
- As predicted by Polyakov, the rigidity parameter m scales with m_0.
  - The rigidity correction becomes dominant in the limit $\beta \to \infty$: different with respect to YM and Abelian Higgs Model.
  - the $U(1)$ model is a perfect laboratory to study the crossover from NG to a rigid string.
- More precise data (or interface free energy) could allow to disentangle boundary and next-order rigidity contribution.
- String broadening behaviour should deviate with respect to NG predicted behaviour: we expect a rigid string to have a constant width at varying sources separation.

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